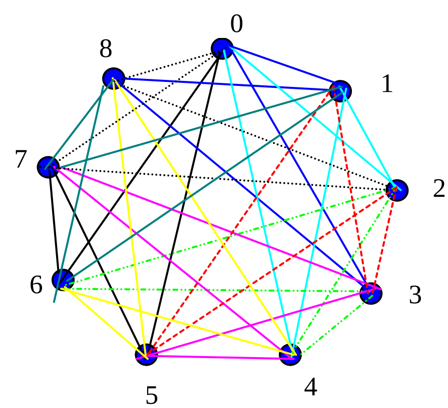
INTRODUCTION TO GRACEFUL GRAPHS



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Acknowledgment

I am deeply indebted to my late supervisor Prof. Jaromir Abrham(1937-1996). He introduced me to the world of graph theory and was always patient, encouraging and resourceful. I have learned very important lessons from him about the research and the academic life.

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TABLE OF CONTENENTS

LIST OF SYMBOLS

LOGIC

$p \Rightarrow q$	The logical implication
$p \Leftrightarrow q$	The logical equivalence
$\forall\, x$	For all x
$\exists\, x$	For some x

SET THEORY

$x \in A$	Element x is a member of set A
$x \notin A$	Element x is not a member of set A
$\lvert A \rvert$	The cardinality of set A
$\lfloor x \rfloor$	The greatest integer less than or equal to the number x
$A \subset B$	A is a subset of B
$A \cup B$	The union of sets A, B
$A \cap B$	The intersection of sets A, B

NUMBERS

Z	The set of integers: {0, 1, -1, 2, -2, 3, -3, … }
N	The set of nonnegative integers: {0, 1, 2, 3, … }
$a \mid b$	a divides b, for a, b $\in$ Z, $a \neq 0$
$\lfloor x \rfloor$	The greatest integer less than or equal to the real number x
$a \equiv b \pmod{n}$	a is congruent to b modulo n

GRAPH THEORY

$G = (V, E)$	G is a graph with vertex set V and edge set E
$T = (V, E)$	T is a tree with vertex set V and edge set E
$G_1 + G_2$	The joint of the two graphs G_1 (V_1, E_1) and G_2 (V_2, E_2)
$G_1 \otimes G_2$	The Cartesian product of G_1 (V_1, E_1) and G_2 (V_2, E_2)
$G_1 \cup G_2$	The disjoint union of the graphs G_1 (V_1, E_1) and G_2 (V_2, E_2)
$G_1 \mid G_2$	A G_1- decomposition of a graph G_2
C_n	C is a cycle of length n

W_n	W is a wheel obtained from the cycle C_n
R_n	R is a crown with 2n edges
H_n	H is a helm with 3n edges
P_n	P is a path or snake of length n
D_n (m)	D is a dragon obtained by joining the end point of path P_m to the cycle C_n
Δ_n-snake	A triangular snake with n blocks
K_n	The complete graph on n vertices
$K_{n1, n2}$	The complete bipartite graph on $n_1 + n_2$ vertices and $n_1 \times n_2$ edges
mK_n	The windmills graph consists of m copies of K_n
Q_n	Q is the n-cube
B_n	B is a book given by $K_{1, n} \otimes K_2$
Z_T	The base of a tree T
T^*	A tree obtained from T by replacing every edge of T by a path of length 2

GRAPH LABELING

$\Psi(V, E)$	A labeling of a graph G (V, E)
$\Psi^{\oplus}(V, E)$	The complementary labeling of a graph G (V, E)
$\Psi^{\lozenge}(V, E)$	The inverse labeling of a graph G (V, E)
$\Psi(v)$	The label of vertex v
$\Psi^{\bullet}(e)$	The label of edge e
$\Phi(V, E)$	The k-graceful labeling of a graph G (V, E)
$\Gamma(V, E)$	The k-sequential labeling of a graph G (V, E)
γ	The special number in an α-labeling of a Graph G (V, E)
N_k (n)	The number of α_k-valuation of P_n

1. INTRODUCTION

A *graph* G = (V,E) consists of two finite sets: V(G), the *vertex set* of the graph, often denoted by just V, which is a nonempty set of elements called *vertices*, and E(G), the *edge set* of the graph, often denoted by just E, which is a set (possibly empty) of elements called *edges*. A graph, then , can be thought of as a drawing or diagram consisting of a collection of vertices (dots or points) together with edges (lines) joining certain pairs of these vertices. Figure 1 provides a graph G = (V,E) with V(G) = { v_1, v_2, v_3, v_4, v_5 } and E(G) = { e_1, e_2, e_3, e_4, e_5, e_6, e_7 }.

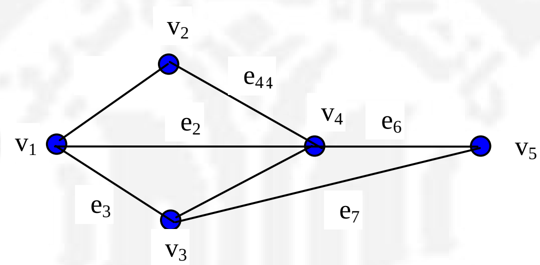


Figure 1: A graph G with five vertices and seven edges

Sometimes we represent an edge by the two vertices that it connects. In Figure 1 we have $e_1 = (v_1, v_2)$, $e_2 = (v_1 ,v_4)$. An edge e of graph G is said to be *incident* with the vertex v if v is an end vertex of e. For instance in Figure 1 an edge e_1 is incident with two vertices v_1 and v_2. An edge e having identical end vertices called a *loop*. In other words, in a loop a vertex v is joined to itself by an edge e. The *degree* of a vertex v, written d(v), is the number of edges incident with v. In Figure 1 we have $d(v_1) = 3$, $d(v_2) = 2$, $d(v_3) = 3$, $d(v_4) = 4$ and $d(v_5) = 2$. If for some positive integer k, d(v) = k for every vertex v of graph G, then G is called *k-regular*.

A graph G is called *connected* if there is a path between every pair of vertices. When there is no concern about the direction of an edge the graph is called *undirected*. The graph in Figure 1 is a connected and undirected graph. Unlike most other areas in Mathematics , the theory of graphs has a definite starting point, when the Swiss mathematician Leonard Euler (1707-1783) considered the problems of the seven Konigsberg bridges. In the early 18th century the city of Konigsberg (in Prussia) was divided into four sections by the Pregel river. Seven bridges connected these regions as shown in Figure 2 (a). Regions are shown by A, B, C, D respectively. It is said that the townsfolk of Konigsberg amused themselves by trying to find a route that crossed each bridge just once (It was all right to come to the same island any number of times).

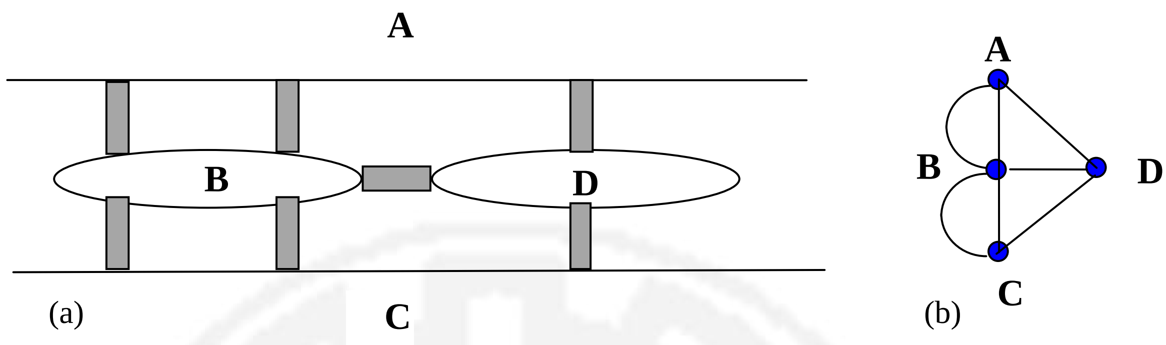


Figure 2: (a) A map of Konigsberg (b) A graph representing the bridges of Konigsberg

Euler discussed whether or not it is possible to have such a route by using the graph shown in Figure 2 (b). He published the first paper in graph theory in 1736 to show the impossibility of such a route and give the conditions which are necessary to permit such a stroll. Graph theory was born to study problems of this type.

Graph theory is one of the topics in an area of mathematics described as *Discrete Mathematics*. The problems as well as the methods of solution in discrete mathematics differ fundamentally from those in continuous mathematics. In discrete mathematics we "count" the number of objects while in continuous mathematics we "measure" their sizes. Although discrete mathematics began as early as man learned to count, it is continuous mathematics which has long dominated the history of mathematics. This picture began to change in twentieth century. The first important development was the change that took place in the conception of mathematics. Its central point changed from the concept of a number to the concept of a set which was more suitable to the methods of discrete mathematics than to those of continuous mathematics. The second dramatic point was the increasing use of computers in society. Much of the theory of computer science uses concepts of discrete mathematics.

Graph theory as a member of the discrete mathematics family has a surprising number of applications, not just to computer science but to many other sciences (physical, biological and social), engineering and commerce.

Some of the major themes in graph theory are shown in Figure 3. Most of these topics have been discussed in text books.

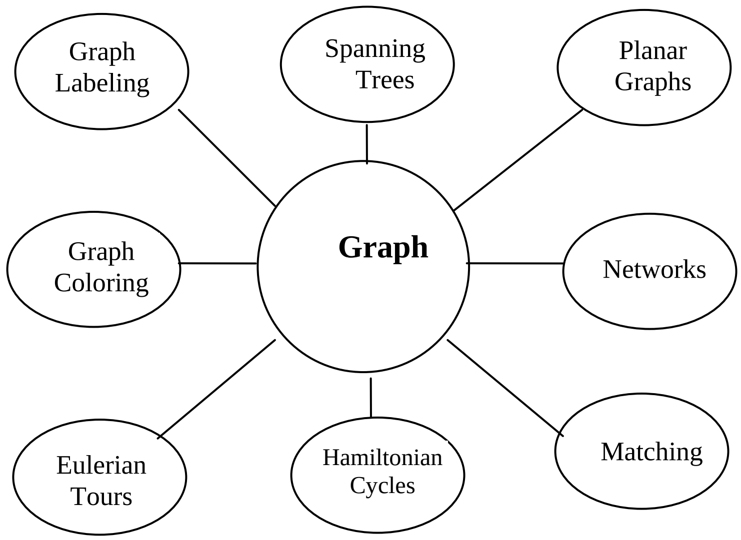


Figure 3: Some topics in Graph Theory

The purpose of this book is to provide some results in a class of problems categorized as *Graph labeling*. Let G be an undirected graph without loops or double connections between vertices. In *labeling* (*valuation* or *numbering*) of a graph G, we associate distinct nonnegative integers to the vertices of G as *vertex labels* (*vertex values* or *vertex numbers*) in such a way that each edge receives a distinct positive integer as an *edge label* (*edge value* or *edge number*) depending on the vertex labels of vertices which are incident with this edge.

Interest in graph labeling began in mid-1960s with a conjecture by Kotzig-Ringel and a paper by Rosa[90]. In 1967, Rosa published a pioneering paper on graph labeling problems. He called a function f a *β-labeling* of a graph G with n edges (Golomb [45] subsequently called such labeling *graceful* and this term is now the popular one) if f is an injection from the vertices of G to the set $\{0, 1, \dots, n\}$ such that, when each edge is labeled with the absolute value of the difference between the labels of the two end vertices, the resulting edge labels are distinct. This labeling provides a sequential labeling of the edges from 1 to the number of edges. Any graph that can be gracefully labeled is a *graceful graph*.

Examples of graceful graphs are shown in Figure 4. Other examples of graceful graphs will be shown in the next chapter.

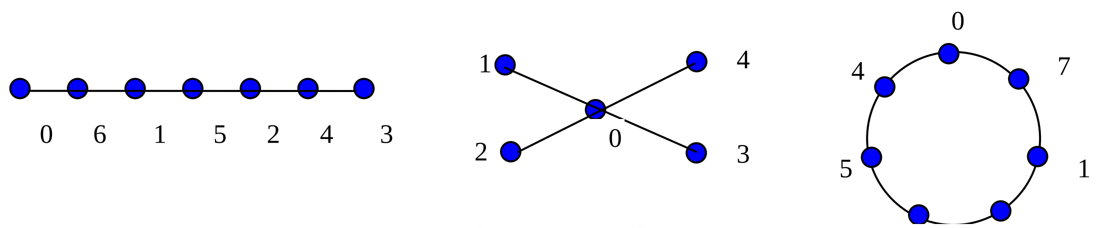


Figure 4: Examples of graceful labeling of graphs

Although numerous families of graceful graphs are known, a general necessary or sufficient condition for gracefulness has not yet been found. Also It is not known if all tree graphs are graceful.

Another important labeling is an *α-labeling* or *α-valuation* which was also introduced by Rosa [90]. An α-valuation of a graph G is a graceful valuation of G which also satisfies the following condition: there exists a number γ $(0 \le \gamma < E(G))$ such that, for any edge $e \in E(G)$ with the end vertices $u, v \in V(G)$,

$$\min \{ \text{vertex label } (v), \text{vertex label } (u) \} \le \gamma < \max \{ \text{vertex label } (v), \text{vertex label } (u) \}$$

It is clear that if there exists an α-valuation of graph G, then G is a bipartite graph. The first graph in Figure 4 is a path with six edges and it has an α-labeling with $\gamma = 3$.

During the past thirty years, over 200 papers on this topics have been appeared in journals. Although the conjecture that all trees are graceful has been the focus of many of these papers, this conjecture is still unproved. Unfortunately there are few general results in graph labeling. Indeed even for problems as narrowly focused as the ones involving the special classes of graphs, the labelings have been hard-won and involve a large number of cases.

Finding a graph that possesses an α-labeling is another common approaches in many papers. The following condition (due to Rosa) is known to be necessary and in the case of cycles also sufficient for a 2-regular graph G = (V,E) to have an α-labeling: $|E(G)| \equiv 0 \pmod 4$. In 1982, Kotzig conjectured that this condition is also sufficient for a 2-regular graph with two components. In 1996, Abrham and Kotzig have shown that this conjecture is valid.

Terms and notation not defined in this book follow that used in [28] and [29].

2. BASIC DEFINITIONS

Let G = (V,E) be a graph with m = $|V|$ vertices and n = $|E|$ edges. By the term graph, we understand a connected, undirected finite graph without loops or multiple edges.

Definition 1: A *labeling* (or *valuation*) of a graph G = (V,E) is a one-to-one mapping Ψ of the vertex set V(G) into the set of non negative integers that induces for each edge $\{u,v\} \in E(G)$ a label depending on the vertex labels Ψ (u) and Ψ(v).

Definition 2: A *graceful labeling* (or *β-valuation*) of a graph G = (V,E) with m = $|V|$ vertices and n = $|E|$ edges is a one-to-one mapping Ψ of the vertex set V(G) into the set { 0,1,2, … ,n } with the following property:
If we define, for any edge e =$\{u,v\} \in E(G)$, the value $\Psi^{\bullet}(e) = |\Psi(u)-\Psi(v)|$ then $\Psi^{\bullet}$ is a one-to-one mapping of the set E(G) onto the set { 1,2, … ,n}.

A graph is called graceful if it has a graceful labeling. The concept of a β-valuation was introduced by Rosa [90] in 1966. Then in 1972 Golomb [45] called such labeling graceful and this name was popularized by mathemagician Martin Gardner [44]. This terminology is now the most commonly used. Let K_n, C_n and T denote respectively a complete graph on n vertices, a cycle of length n and a tree, then Figure 5 gives us graceful labelings of K_3, C_4, tree T and the Petersen graph:

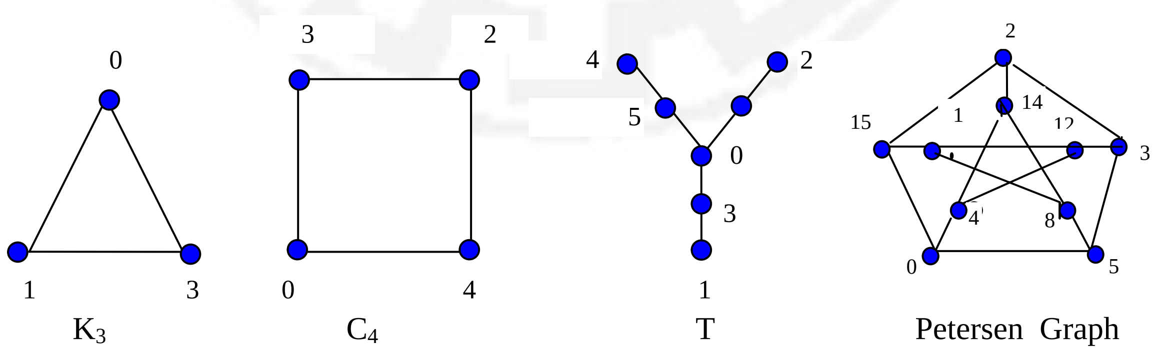


Figure 5: Some graceful graphs

Not all graphs are graceful, for example C_5 and K_n for $n > 4$ are not graceful [45]. A given graph may have several distinct graceful labelings as it is shown in Figure 6:

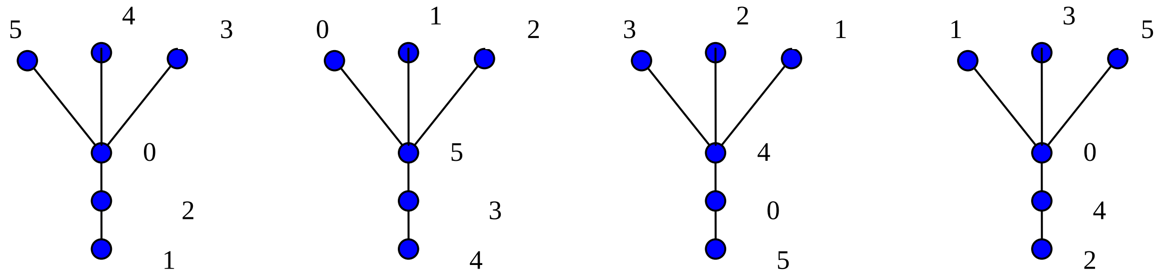


Figure 6: Several graceful labelings of a graph

Sheppard [94] has shown that there are exactly n! gracefully labeled graphs with n edges. Erdos [40] in an unpublished paper proved that most graphs are not graceful. In Figure 7, we present four non-graceful graphs, see [21] for a proof that K_5 and C_5 are not graceful, see [22] for the third graph, and [24] for the last graph:

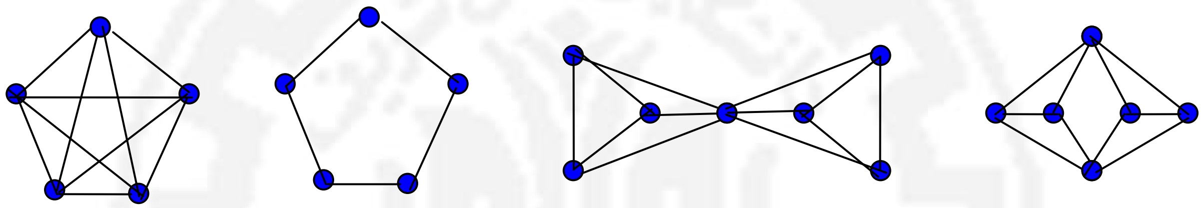


Figure 7: Some non-graceful graphs

Notice that a subgraph of a graceful graph need not be graceful. For example C_5 is a subgraph of a Petersen graceful graph but C_5 is not graceful. In [90] Rosa also defined an α-labeling of a graph, a graceful labeling with an additional property, as follows:

Definition 3: An *α-labeling* (or *α-valuation*) of a graph $G = (V,E)$ is a graceful labeling of G which satisfies the following additional condition:

There exists a number γ ($0 \le \gamma \le |E(G)|$) such that, for any edge $e \in E(G)$ with the end vertices $u,v \in V(G)$, it has $\min[\Psi(u),\Psi(v)] \le \gamma < \max[\Psi(u),\Psi(v)]$.

For instance C_4 in Figure 5 has an α-valuation with $\gamma = 2$ and in Figure 6 the first three trees have an α-labeling with $\gamma_1 = 1$, $\gamma_2 = 3$, $\gamma_3 = 3$ but the last tree does not.

Definition 4: The values of an α-labeling Ψ which are $\leq \gamma$ will be referred as "*small values*" and the remaining values of Ψ as the "*large values*" of a given α-valuation.

The small values of an α-valuation of C_4 in Figure 5 are {0,2} and the large values are {3,4}. The definitions 3 and 4 imply that a graph with an α-valuation is necessarily bipartite and therefore cannot contain a cycle of odd length.
In 1992, Gallian [43] weakened the condition for an α-valuation by the following definition:

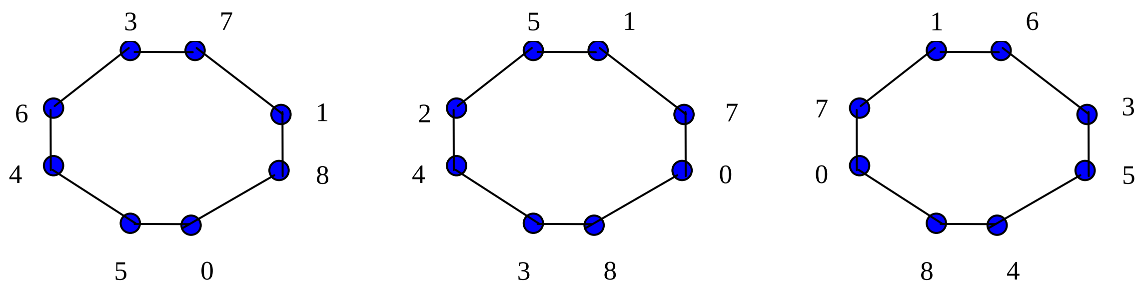
Definition 5: A *weakly α-labeling* (or a *weakly α-valuation*) of a graph G = (V,E) is a graceful labeling of G which satisfies the following additional condition:
There exists s number γ^* $(0 \leq \gamma^* \leq |E(G)|)$ such that , for any edge $e \in E(G)$ with the end vertices $u,v \in V(G)$, it has $\min[\Psi(u),\Psi(v)] \leq \gamma^* \leq \max[\Psi(u),\Psi(v)]$.

Therefore the condition for weakly α-labeling allows the graph to have an odd cycle. For example in Figure 5, K_3 has no α-labeling but it has a weakly α-labeling with $\gamma^* = 1$.
Notice that while γ in an α-valuation is the lesser of the two labels whose difference is 1, γ^* in a weakly α-labeling may be either of the two labels whose difference is one. Furthermore if a graph has a weakly α-labeling with γ^* then the vertex labeled γ^* must be on every odd cycle.
Now we should mention two transformations of α-labeling (graceful labeling) which are sometimes useful:

Definition 6: If Ψ is an α-labeling (or a graceful labeling) of a graph G = (V,E) with $n = |E|$ edges then the valuation $\Psi^{\oplus}$ defined by $\Psi^{\oplus}(v) = n - \Psi(v)$ for all $v \in V(G)$ is again an α-labeling (or a graceful labeling) of G and called *complementary labeling* (or *complementary valuation)* to Ψ.

Definition 7: If Ψ is an α-labeling of a graph G = (V,E) with $n = |E|$ edges and if we put $\Psi^{\Diamond}(v) \equiv \gamma - \Psi(v) \pmod{n+1}$ for every $v \in V(G)$; $\Psi^{\Diamond}(v) \subset \{0, 1, \ldots, n\}$ then $\Psi^{\Diamond}$ is again an α-labeling of G and called *inverse labeling* (or *inverse valuation)* to Ψ.

In Figure 8, an α-labeling, a complementary labeling and an inverse labeling of C_8 are shown:



Ψ: α-labeling of C_8 $\Psi^{\oplus}$: complementary labeling of C_8 $\Psi^{\lozenge}$: inverse labeling of C_8

Figure 8: α-labeling and its complementary and inverse valuations of C_8

Although we focus on graceful labeling and α-labeling in this book, we also discuss important variations of graceful labeling as follows:

Definition 8: A k-graceful labeling of a graph G = (V,E) with n = $|E(G)|$ edges is a one-to-one mapping f of the vertex set V(G) into the set {0,1,2, …,n+k-1} such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is {k,k+1,k+2, … ,n+k-1}.

The concept of k-graceful labeling was introduced simultaneously by Slater [98] and by Maheo and Thuillier [84]. Now let us define a *wheel* W_n as a graph obtained from the cycle C_n by adding a new vertex and edges joining it to all the vertices of the cycle; n is assumed to be at least three. In Figure 9, a 7-graceful labeling of C_{15} and a 3-graceful labeling of W_7 are shown

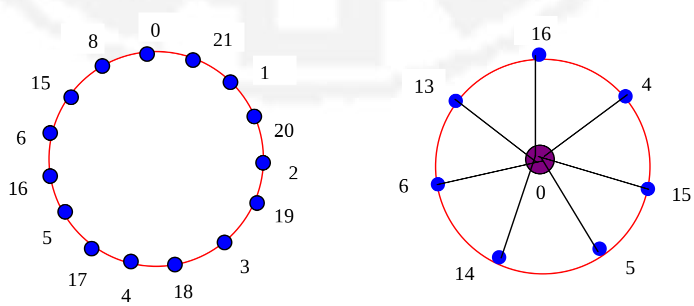


Figure 9: 7-graceful labeling of C_{15} and 3-graceful labeling of W_7

It is obvious that the standard definition of graceful labeling corresponds to a 1-graceful labeling. If there exists an α-labeling Ψ of graph $G = (V,E)$, then for any $k \geq 1$ graph G is k-graceful with the labeling Φ given as follows:

$$\Phi(v) = (\ \Psi(v) \text{ if } \Psi(v) \leq \gamma, \ \Psi(v)+k-1 \ \text{ if } \Psi(v) > \gamma, \ v \in V(G), \ v \in V(G) \ )$$

In Figure 10, an α-valuation of C_4 is changed to a 6-graceful labeling by using the above transformation:

An α-valuation of C_4 6-graceful labeling of C_4

Figure 10: Transformation of an α-labeling to k-graceful labeling for C_4

Graphs that are k-graceful for all k are sometimes called *arbitrarily graceful* [40]. Ng [86] has shown that an α-valuation is properly stronger than k-graceful for all k.

In the following method of labeling; instead of using a function from the vertices of a graph to a set of labels, we will use a function from the vertices and edges to a set of labels:

Definition 9: A *k-sequential labeling* of a graph $G = (V,E)$ with $n = |E(G)|$ edges and $m = |V(G)|$ vertices is a one-to-one function Γ from $V(G) \cup E(G)$ to $\{k, k+1, k+2, \ldots, n+m+k-1\}$ such that for each edge $e=\{u,v\} \in E(G)$, one has $\Gamma(e) = |\Gamma(u) - \Gamma(v)|$.

A graph G admitting a k-sequential labeling is called a "*k-sequential graph*". If G is a 1-sequential graph, it is called a *simply sequential graph*. Simply sequential and sequential graphs were first defined in [20]. The wheel W_6 and the cycle C_4 in Figure 11 are simply sequential but graph G_1 is 4-sequential:

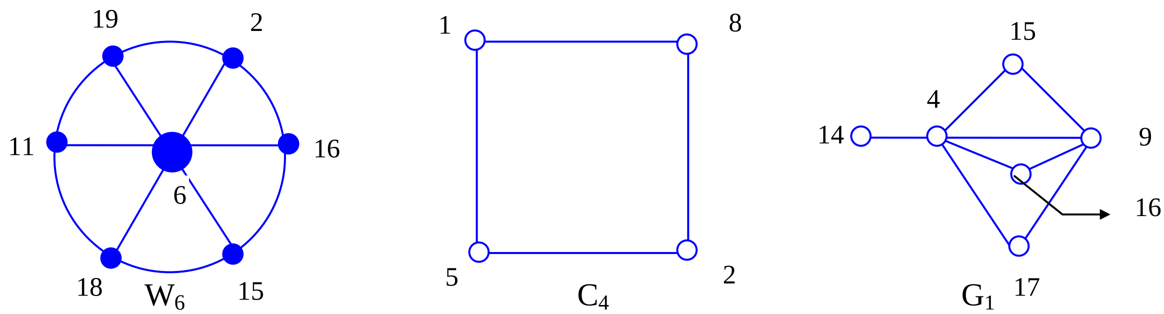


Figure 11: Examples of simply sequential graphs and a 4-sequential graph

By considering the similarities in definitions of graceful graphs and sequential graphs we should not be surprised if there is a connection between these graphs. Now before we explain the relation between these two kinds of labeling, let us define the following operation in two graphs:

Definition 10: The *join of the two graphs* $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ denoted by $G_1+ G_2$, is defined as $V(G_1+ G_2) = V_1 \cup V_2$; $V_1 \cap V_2 = \varnothing$ and $E(G_1+ G_2) = E_1 \cup E_2 \cup I$ where $I = \{ (v_1,v_2): v_1\in V_1, v_2\in V_2 \}$. Thus I consists of edges which join every vertex of G_1 to every vertex of G_2.

In [97], Slater proved that a graph G is simply sequential if and only if the join of G and an isolated vertex i.e. G + v has a graceful valuation Ψ with $\Psi(v) =0$.

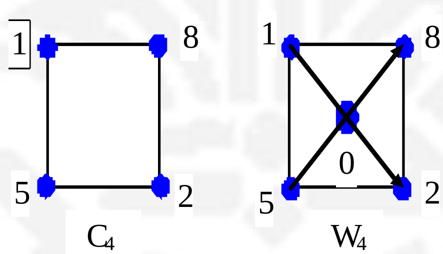


Figure 12: Corresponding 1-sequential labeling of C_4 and graceful labeling of W_4

In Figure 12, we see that C_4 is simply sequential, then C_4 + v or in the other words W_4 has a graceful labeling Ψ with Ψ (v) = 0 as illustrated in Figure 12 [20].

In [15], Acharya has shown a fundamental link between k-graceful and k-sequential graphs by generalizing the Slater result:

A graph G is k-sequential if and only if G +v has a k-graceful labeling Ψ with Ψ(v) =0.

For example in Figure 13, a 3-graceful labeling W_7, shown before in Figure 9, is transformed to a 3-sequential labeling of C_7.

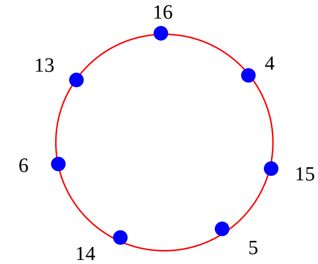


Figure 13: 3-sequential labeling of C_7

The methods of labeling of a graph have been extended rapidly in the last ten years. A number of new methods of labeling have been investigated such as *Cordial Labeling* [30], *Harmonious Labeling* [48], *Elegant Labeling* [31], *Prime Labeling* [75], and *Sum Labeling* [50]. In two excellent surveys by Gallian [40,41], he has summarized much of what is known about each kind.

Now, we will focus on graceful labeling and its variations and summarize the results obtained to date about these kinds of labeling in different classes of graphs. We discuss graceful labeling and its valuation on the following classes of graphs (see also [40, 41]):

1. Cycle- related graphs with one component
2. Complete graphs
3. Cartesian-related graphs
4. Tree-related graphs
5. Disjoint union of graphs

3. CYCLE-RELATED GRACEFUL GRAPHS WITH ONE COMPONENT

The following necessary condition for gracefulness of a graph G = (V,E) with $m = |V(G)|$ and $n = |E(G)|$ comes directly from the definition 2.2:

Lemma 1 [90]: If G is a graceful graph then $m \leq n+1$.

It is clear that the above lemma is satisfied for every connected graph. Using this condition we can rule out the existence of a graceful labeling for some disconnected graphs, for instance, 1-regular graphs with $n > 1$.

A connected graph G is called *Eulerian* if $n > 0$ and the degree of every vertex of G is even. A necessary condition for the existence of a graceful labeling of an Eulerian graph G is proved by Rosa [90]:

Theorem 1 [90]: If G is a graceful Eulerian graph then $n \equiv 0$ or 3 (mod 4).

In this theorem, an Eulerian graph is any graph in which the degree of each vertex is even; it does not have to be connected.

For example, K_5 and C_5 in Figure 7 are Eulerian, but they have 10 and 5 edges respectively and thus by the above theorem they are not graceful.

A generalization of Rosa's theorem for k-graceful Eulerian graphs is as follows:

Theorem 2.2 [15]: If an Eulerian graph G = (V,E) is k-graceful then either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ when k is even or $n \equiv 3 \pmod 4$ when k is odd.

For cycle C_n, the necessary condition in theorem 2.1 is also sufficient:

Theorem 3 [90]: The cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4).

Rosa also proved the following result:

Theorem 4 [90]: The cycle C_n has an α-labeling if and only if $n \equiv 0 \pmod 4$.

Maheo and Thuillier [84] have generalized this result as follow:

Theorem 5 [84]: The cycle C_n is k-graceful if and only if either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ where k is even and $k \leq (n-1)/2$ or $n \equiv 3 \pmod 4$ where k is odd and $k \leq (n-1)/2$.

We also know that:

Theorem 6 [20]: The cycle C_n is 1-sequential.

According to theorem 6 and the connection between 1-sequential and graceful graphs, we can conclude that all wheels are graceful:

Theorem 7 [53]: The wheel W_n is graceful for all $n \geq 3$.

The following theorem and conjecture are due to Maheo and Thuillier:

Theorem 8 [84]: W_{2k+1} is k-graceful for any $k \geq 1$.

Conjecture 1 [84]: W_{2k} is k-graceful with $k \neq 3,4$.

A *crown* R_n is formed by adding to the n points $v_1, v_2, v_3, \dots, v_n$ of a cycle C_n, n more pendant points $u_1, u_2, u_3, \dots, u_n$ and n more lines (u_i, v_i), $i = 1,2,3, \dots, n$ for $n \geq 3$. Frucht [35] has proved the following theorem:

Theorem 9 [35]: R_{2n} is graceful for any $n \geq 3$.

We know that a graph admitting an α-labeling must be bipartite and, as such, can not contain cycles of odd length. It follows that wheels can not have an α-labeling since they contain triangles as subgraphs. For analogous reason, crowns can not have α-labeling if n is odd. For even values of n, Frucht has offered the following conjecture:

Conjecture 2 [35]: If $n \equiv 0 \pmod 2$ then R_n has an α-labeling.

In Figure 10, we can see a graceful labeling for R_5 and an α-labeling for R_6:

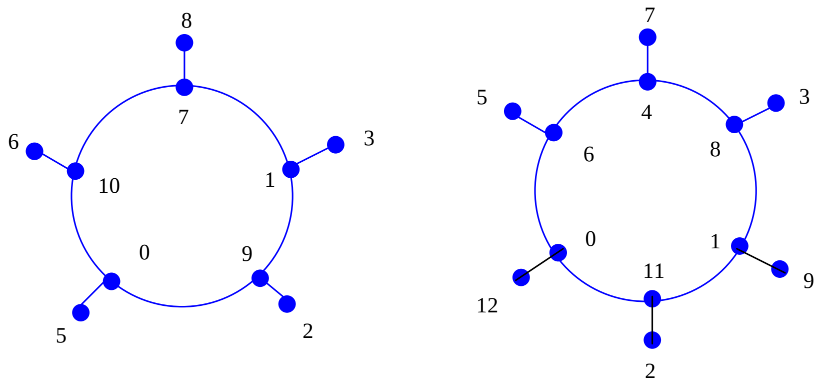


Figure 14: Graceful labeling of R_5 and an α-labeling of R_6

A *helm* H_n, $n \geq 3$, is the graph obtained from a crown R_n by adding a new vertex joined to every vertex of the unique cycle of the crown. Ayel and Favaron [18] proved that:

Theorem 10 [18]: The helm H_n is graceful for every $n \geq 3$.

For example, Figure 15 shows that H_5 is graceful:

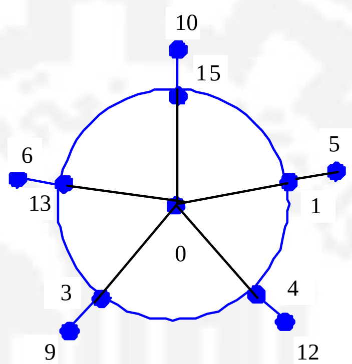


Figure 15: Graceful labeling of H_5

A *chord* of a cycle is an edge joining two otherwise non adjacent vertices of a cycle. Bondendiek [21] conjectured that any cycle with a chord is graceful. This conjecture has been proved by Delorme et al.:

Theorem 11 [34]: The graph consisting of a cycle plus a chord is graceful.

Let P_k be a path with k edges and k+1 vertices (as we can see later the term *snake* is also used in this case). Koh and Yap defined a *cycle with a P_k-chord* as a cycle with a path P_k

joining two nonconsecutive vertices of the cycle. They proved that these graphs are graceful when k =2. Thereafter Punnim and Pabhapate proved the general case k ≥3.

Theorem 12 [65, 88]: A cycle with a P_k-chord is graceful for all k ≥1.

In 1990, Zhi-Zheng generalized the above theorem by proving the following result:

Theorem 13 [104]: Apart from four exceptional cases, simple graphs consisting of three independent paths joining two vertices are graceful.

Examples of graceful labeling of cycles with a P_1-chord and P_3-chord can be seen in Figure 16 :

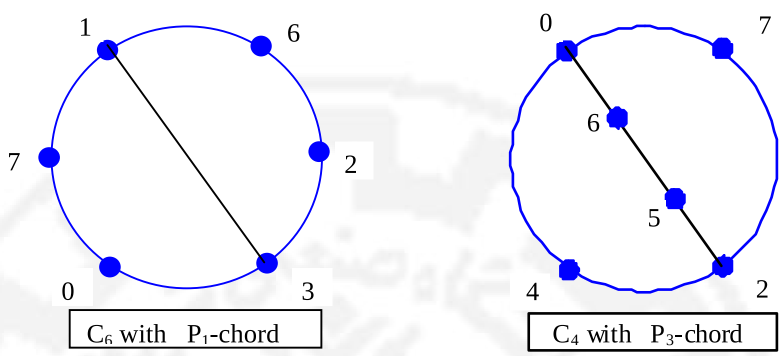


Figure 16: Examples of graceful labeling of cycles with P_k-chord

Koh et al. [40, 64] also introduced the concept of a *cycle with k-consecutive chords*. A cycle with k-consecutive chords is a graph formed from a cycle by joining a cycle vertex v to k consecutive vertices of the cycle in such a way that v is not adjacent to any of these. Koh and others proved the following result about this kind of graph:

Theorem 14 [61,64,65]: A cycle C_n with k-consecutive chords is graceful if k =2, 3, n-3.

A *dragon* D_n (m) is a graph obtained by joining the end point of path P_m to the cycle C_n. Truszcynski has proved the following theorem related to dragons:

Theorem 15 [103]: The dragon $D_n(m)$ is graceful for n ≥ 3, m ≥1.

The following conjecture is also due to Truszcynski:

Conjecture 3 [103]: All graphs with a unique cycle are graceful except C_n, $n \equiv 1,2 \pmod 4$.

Figure 17 shows two graceful graphs: a cycle C_8 with twin chords (or 2-consecutive chords) and a dragon D_5 (3):

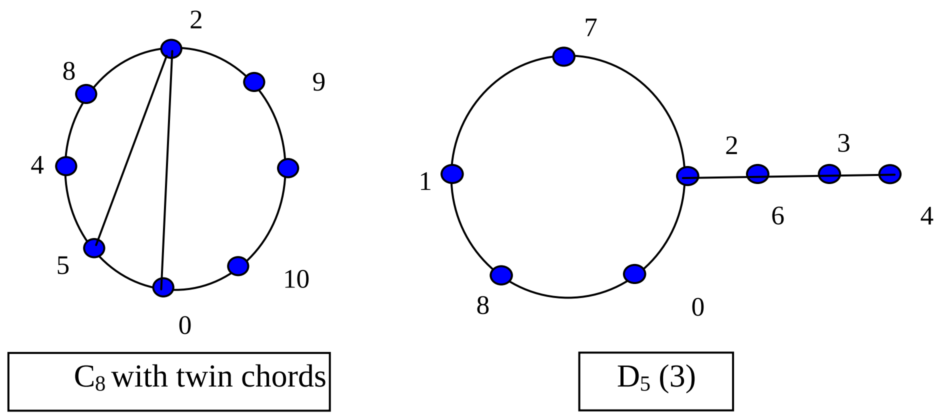


Figure 17: Graceful labeling of a cycle with twin chords and a dragon

Rosa [92] has defined a *triangular snake* (or *Δ-snake*) as a connected graph in which all blocks are triangles and the block-cut-point graph is a path. For definitions of block and block-cut-point graph see [28]. Let Δ_n-snake be a Δ snake with n blocks. Since a Δ_n-snake is an Eulerian graph, according to theorem 2.1 it can only be graceful if $3n \equiv 0$ or $3 \pmod 4 \Rightarrow n \equiv 0$ or $1 \pmod 4$. Moulton verified that this result is also sufficient:

Theorem 16 [85]: Every Δ_n-snake is graceful if and only for $n \equiv 0$ or $1 \pmod 4$.

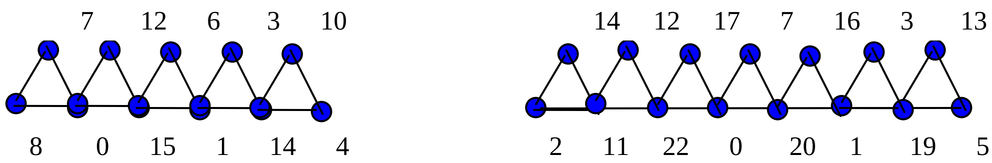
In order to deal with other cases, Moulton also defined a new concept as follows:

Definition 11: An *almost graceful labeling* of a graph G = (V,E) with $n = |E(G)|$ and $m = |V(G)|$ is a one-to-one mapping *f* of the vertex set V(G) into the set $\{0,1,2, \ldots, n-1\} \cup \{n \text{ or } n+1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of the adjacent vertices is $\{1,2,3, \ldots, n-1\} \cup \{n \text{ or } n+1\}$.

Notice that the above definition includes graceful labeling as special case. Next Moulton has strengthened the theorem 2.16 as follows:

Theorem 17 [85]: Every Δ_n-snake for $n \equiv 2$ or $3 \pmod 4$ is almost graceful.

The graceful labeling of Δ_5-snake and an almost graceful labeling of Δ_7-snake are shown in Figure 18:



Δ_5-snake Δ_7-snake

Figure 18: Graceful labeling of Δ_5-snake and an almost graceful labeling of Δ_7-snake

Another class of cycle related graphs is that the disjoint union of cycles. Some recent results for this class of graphs will be summarized in next section.

4. COMPLETE GRACEFUL GRAPHS

In Figures 5 and 7 it was shown that K_3 is graceful but K_5 is not. The following result will answer the question of the gracefulness of the complete graphs:

Theorem 18 [45]: K_n is graceful if and only if $n \leq 4$.

From the above theorem and the relation between graceful labeling and k-sequential labeling it follows that K_n is 1-sequential if and only if $n \leq 3$. Furthermore Slater proved that for $n \geq 2$, K_n is not k-sequential for all $k \geq 2$. Therefore based on the relationship of k-sequential graphs we can conclude that no complete graph K_n is k-graceful for $k \geq 2$, $n \geq 3$.

The *complete bipartite graph* $K_{a,\, b}$ is the graph with $m = a + b$ vertices and $n = a \times b$ edges, obtained by connecting each of the "a" vertices with each of the "b" vertices in all possible ways. For this class of graphs we have the following result proved by Rosa and Golomb:

Theorem 19 [45,90]: The complete bipartite graph $K_{n1,\, n2}$ has an α-valuation for all n_1, $n_2 \geq 1$.

The graceful labeling of $K_{3,\, 3}$ is shown in Figure 19:

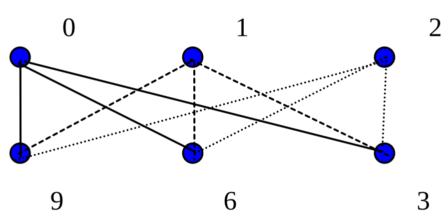


Figure 19: A graceful labeling of $K_{3,3}$

By Kuratowski's theorem [28] we know that a graph is nonplanar if and only if it contains a subgraph that is homomorphic to either K_5 or $K_{3,3}$. Since by considering theorems 18 and 19 K_5 is not graceful but $K_{3,3}$ is, we may conclude that planarity is unnecessary and insufficient for gracefulness.

Since $K_{n1,\, n2}$ has an α-labeling it is k-graceful too. The graph $K_{1,n}$ is known as *star*. Slater showed the following result about stars:

Theorem 20 [98]: The star $K_{1,n}$ is k-sequential if and only if k divides n.

Then Acharya proved that:

Theorem 21 [15]: $K_{n,n}$ is n-sequential for all $n \geq 1$.

In Figure 20, we have shown the 3-sequential labeling for a star $K_{1,6}$ and a bipartite complete graph $K_{3,3}$:

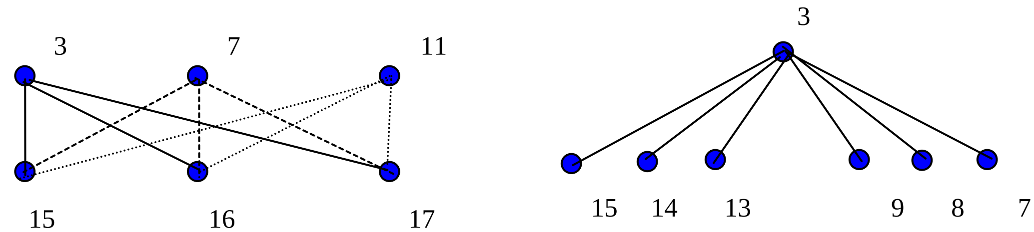


Figure 20: 3-sequential labeling for $K_{3,3}$ and $K_{1,6}$

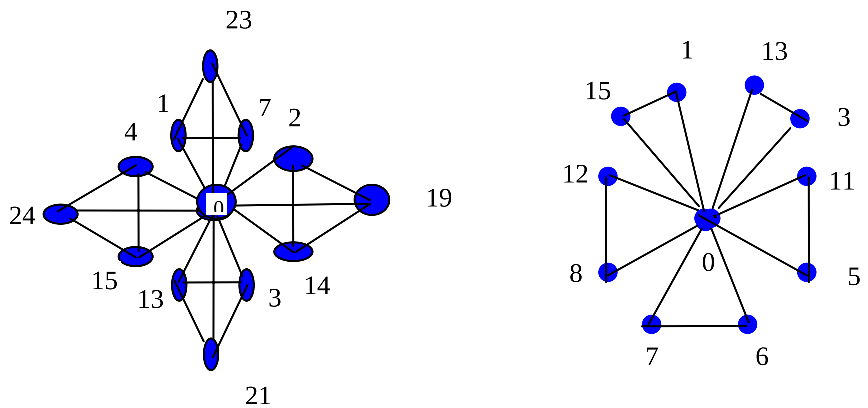
Windmill graphs m K_n ($n \geq 3$) are the family of graphs consisting of m copies of K_n with a vertex in common. Let us call the case n = 3, the graph consisting of mK_3's with one vertex in common, a *Dutch m-windmill*. The graceful labeling of this case was solved by Bermond et al. As follows:

Theorem 22 [22]: The Dutch m-windmill is graceful if and only if $m \equiv 0$ or 1 (mod 4).

For n = 4 we have mK_4's with exactly one vertex in common. It was proposed in 1976 to call this kind of graph a *French m-windmill*. The following conjecture is still an open problem although it is known to be true for $4 \leq m \leq 32$ [22,56]:

Conjecture 4 [22]: The French m-windmill is graceful if $m \geq 4$.

Figure 21 shows a graceful Dutch 5-windmill and a graceful French 4-windmill:



French 4-windmill Dutch 5-windmill

Figure 21: Graceful labeling of French 4-windmill and Dutch 5-windmill

Bermond et al. Also proved that a necessary condition for mK_n to be graceful is that $n \leq 5$. For $n = 5$ the further necessary condition is as follow:

Theorem 23 [22]: If mK_5 is graceful then m is even.

We know that $2K_5$ and $4\ K_5$ are not graceful but $6K_5$ and $8K_5$ are [64]. Let us now use the notation mK_n^r for the graph consisting of m copies of K_n with a K_r in common. The following problem raised by Bermond:

Problem 1 [21]: For which values of m, n and r is the graph mK_n^r graceful ?

In fact the case r = 1 discussed above is a special case of problem 1. The problem 1 has been solved only for the following minor cases when r > 1:

1. $mK_3^{\,2}$ is graceful for all $m \geq 1$ [62].
2. $mK_4^{\,2}$ is graceful for all $m \geq 1$ [33].
3. $mK_4^{\,3}$ is graceful for all $m \geq 1$ [62].
4. $mK_n^{\,r}$ is graceful only if $m \equiv E_{i,\,j}$ (mod 4) where $n \equiv i$ (mod 8), $r \equiv j$ (mod 8) and $E_{i,j}$ is an entry in the i th row and j th column of Table 2.1 [64] (Notation ♣ indicates that there is no graceful labeling for that case.):

i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1,3	0,1,2,3	1	0,1	1,3	1,3	1	1,2
3	1	0,1	1,3	0,1,2,3	1	1,2	1,3	1,3
5	♣	0,2	3	0,3	♣	♣	3	2,3
7	3	0,3	♣	0,2	3	2,3	♣	♣

Table 1: Possible values of $E_{i,j}$ in case 4

For example mK_7^6, mK_{13}^4 and mK_{15}^{10} are not graceful for all $m \geq 1$ but $5K_9^6$, $7K_{21}^{10}$ and $8K_{23}^{17}$ are graceful. In Figure 22 the graceful labeling of $3K_4^2$ and $3K_4^3$ are shown:

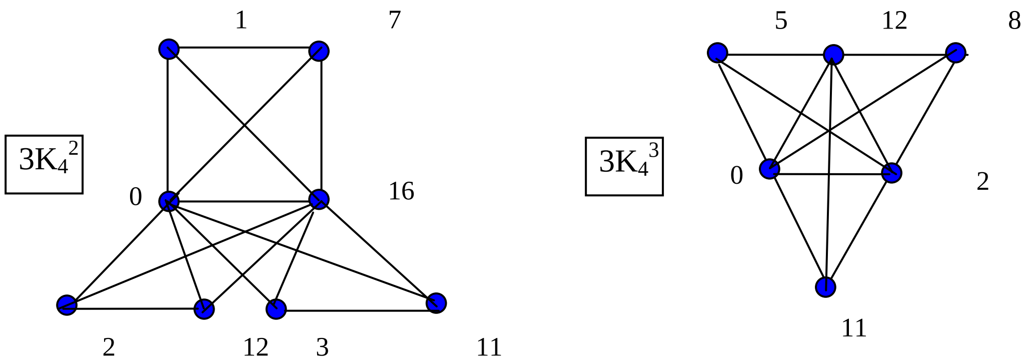


Figure 22: Graceful labeling of $3K_4^2$ and $3K_4^3$

5. CARTESIAN PRODUCT GRACEFUL GRAPHS

Definition 12: A *Cartesian product* of two subgraph G_1 and G_2 is the graph $G_1 \otimes G_2$ such that its vertex set is a Cartesian product of $V(G_1)$ and $V(G_2)$ i.e. $V(G_1 \otimes G_2) = V(G_1) \otimes V(G_2) = \{ (x,y) \mid x \in V(G_1), y \in V(G_2) \}$ and its edge set is defined as $E(G_1 \otimes G_2) = \{ ((x_1,x_2),(y_1,y_2)) \mid x_1 = y_1 \text{ and } (x_2,y_2) \in E(G_2) \text{ or } x_2 = y_2 \text{ and } (x_1,y_1) \in E(G_1) \}$.

For example the *n-cube* Q_n is defined by $Q_1 = K_2$ and $Q_{n+1} = Q_n \otimes K_2$. Q_1, Q_2 and Q_3 are shown in Figure 23:

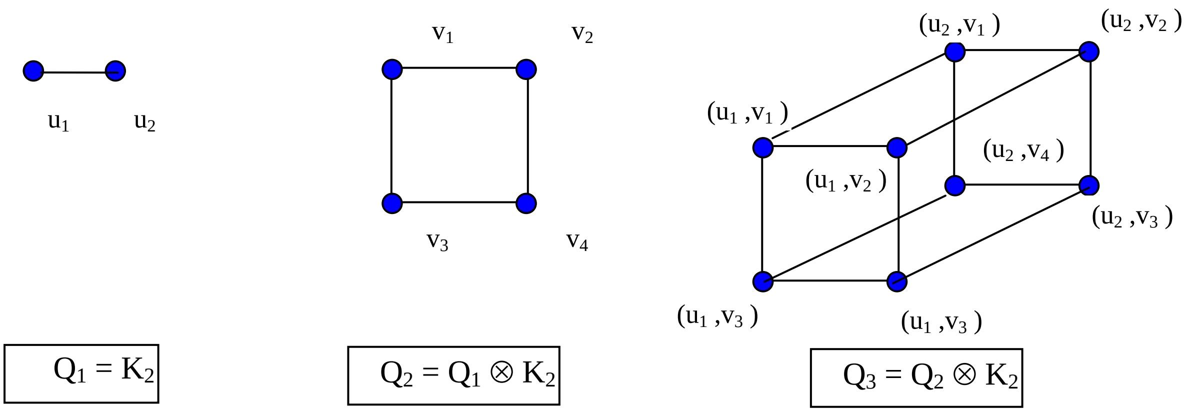


Figure 23: Construction of Q_1, Q_2 and Q_3

Numerous variations of graphs that are Cartesian products have been investigated for graceful labeling. Here we discuss the major results on this topic:

Theorem 24 [69,83]: Q_n has an α-valuation for all $n \geq 1$.

Jungreis and Reid have investigated the existence of α-labeling for a variety of graphs of the form $P_m \otimes P_n$, $C_m \otimes P_n$, and $C_m \otimes C_n$ where P_n is a path on n vertices, and C_n is a cycle on n vertices ($n > 3$). Let us define graphs of the form $P_m \otimes P_n$, $C_m \otimes P_n$,and $C_m \otimes C_n$ as *planar grids*, *prisms* (or *cylindrical grids*), and *torus grids* respectively:

The progress to date [59] in planar grids, prisms and torus grids is summarized in Table 2 below. The entry YES (or NO) shows that the labeling is possible (or impossible). The number in [] refers to the other references in addition to Jungreis and Reid; the question mark ? means that the case is still an open problem. Note that all negative results for α-valuations follow simply because that graph is not bipartite:

Name	Graph	Graceful Labeling	α-labeling
Planar Grids	$P_m \otimes P_n$	YES [16]	YES [16]
Prisms	$C_{2m} \otimes P_{2n}$	YES [36]	YES
	$C_{4m} \otimes P_{2n+1}$	YES	YES
	$C_{2m+1} \otimes P_n$	YES for $2 \le n \le 12$ [36,56], Otherwise ?	NO
	$C_{4m+2} \otimes P_{2n+1}$	?	?
Torus Grids	$C_{4m} \otimes C_{2n}$	YES	YES
	$C_{4m} \otimes C_{2n+1}$	?	NO
	$C_{2m+1} \otimes C_{2n+1}$	NO	NO
	$C_{4m+2} \otimes C_{2n+1}$	?	NO
	$C_{4m+2} \otimes C_{4n+2}$	?	?

Table 2: Recent results in labeling of different variations of grids

Figure 24 gives α-labeling for planar grid $P_4 \otimes P_5$ and graceful labeling for prism $C_5 \otimes P_2$ and torus grids $C_4 \otimes C_6$:

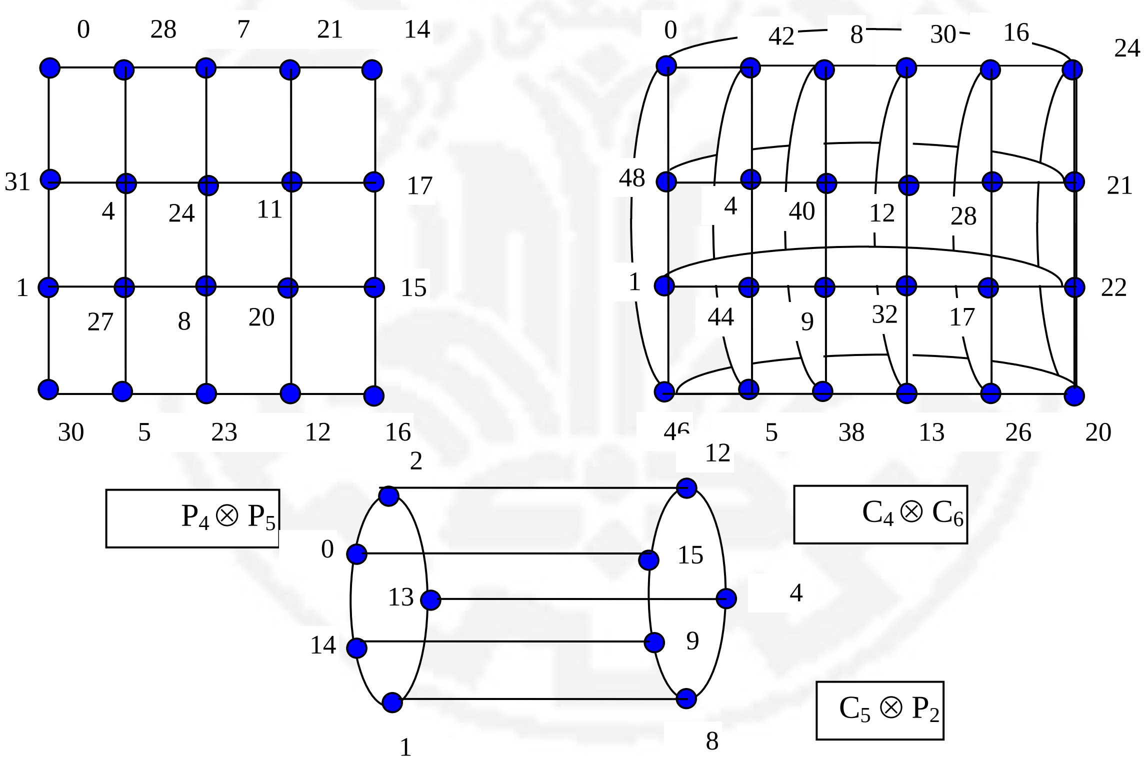


Figure 24: Examples of labeling for Cartesian product graphs

A *book B_n* is the graph $K_{1,n} \otimes K_2$ where $K_{1,n}$ is the star with n edges. The following theorem is due to Maheo:

Theorem 25 [83]: The book B_{2n} has an α-labeling for all $n \geq 1$.

Maheo also conjectured that the books B_{4n+1}, or in the other words the union of $(4n+1)C_4$ having one edge in common, were also graceful. This conjecture was verified by Delorme:

Theorem 26 [33]: The book B_{4n+1} is graceful for all $n \geq 1$.

Gallian and Jungreis [42] have generalized this class of graph by defining a *stacked book $SB_{n,m}$* as a graph of the form $K_{1,n} \otimes P_m$. They proved the following theorem in this case:

Theorem 27 [42]: The stacked book $SB_{2n,m}$ is graceful for all $m, n \geq 1$.

The graceful labeling of the stacked book $SB_{2n+1,m}$ is still an open problem. In Figure 25 an α-labeling of the book B_6 and a graceful labeling of the book B_5 and stacked book $SB_{2,3}$ are shown:

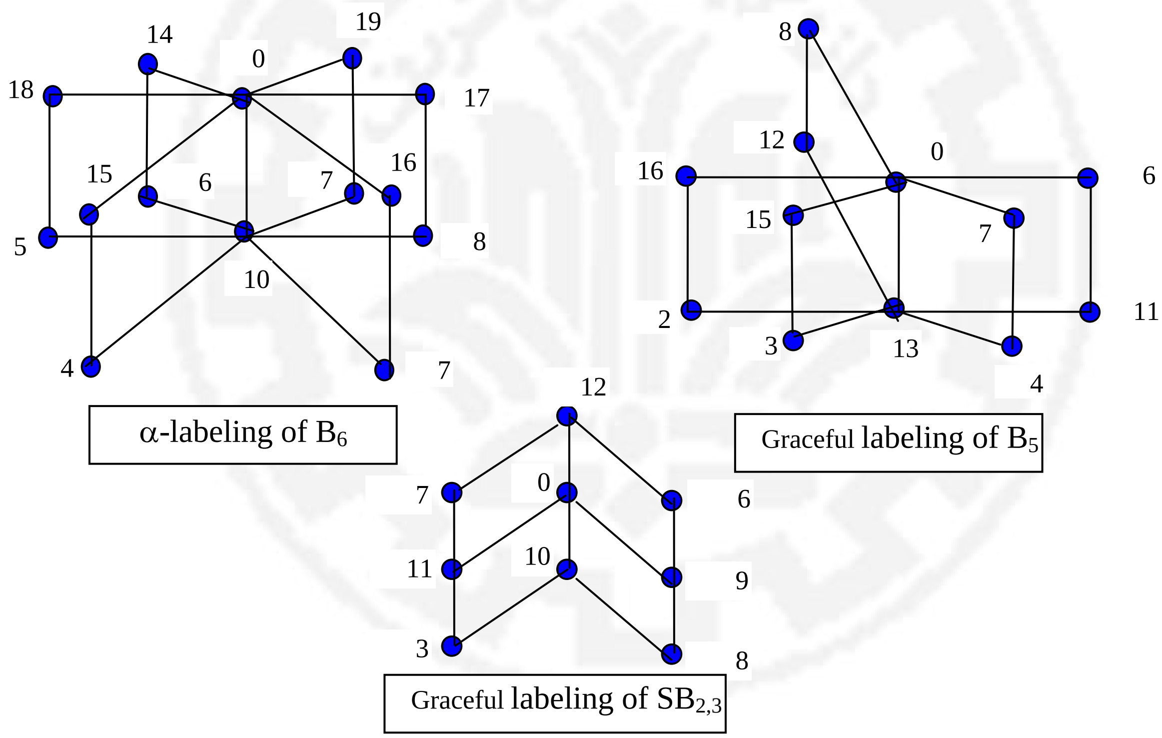


Figure 25: Examples of graceful labeling and α-labeling of books and stacked books

Now let $Q_n(G) = G \otimes \underbrace{K_2 \otimes K_2 \otimes \dots \otimes K_2}_{(n\text{-}1)\text{ times}}$ denote the graph of n-dimensional G-cube. Balakrishnan and Kumar have proved that Q_n (G) has an α-valuation for the special cases of G:

Theorem 28 [19]: $Q_n(G)$ has an α-labeling if G $=K_{3,3}$, $K_{4,4}$, or P_k for all $n \geq 1$, $k \geq 2$.

In Figure 26, an α-labeling for Q_2 $(K_{3,3})$ is shown:

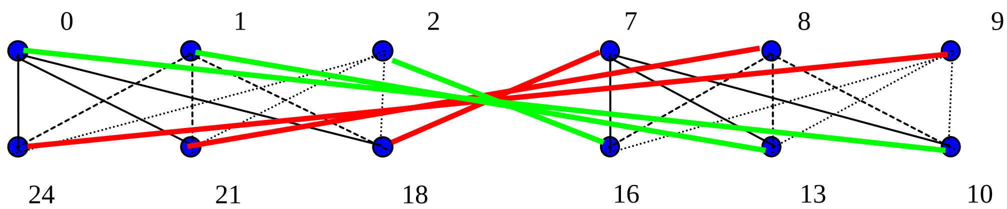


Figure 26: An α-valuation for $Q_2(K_{3,3})$

6. TREE-RELATED GRACEFUL GRAPHS

The most well known problem in graph labeling emanates from a problem formulated by Ringle and a subsequent conjecture by Kotzig:

Conjecture 5 [89,90]: All trees are graceful.

Despite massive efforts, almost 100 papers, this conjecture has not been proved yet; however, many classes of trees have been shown to be graceful. Rosa [90] proved that not all trees admit an α-labeling although Kotzig [67] proved that almost all trees have an α-labeling. For instance the tree T in Figure 5 is one of rare examples of trees with no α-labeling.

From now on let us consider P_n as a snake (or path) with n edges. A subgraph Z_T is called *the base of a tree* T when Z_T is obtained from T by omitting all its end vertices (vertices of degree one) and end edges. If T is not a snake but Z_T is, then T is called a *caterpillar*. The following result is due to Rosa:

Theorem 29 [90]: If T is a snake or caterpillar then T has an α-labeling.

It is obvious that every snake or caterpillar is also k-graceful for all $k \geq 1$. Examples of an α-labeling for a snake P_5 and a caterpillar can be seen in Figure 27:

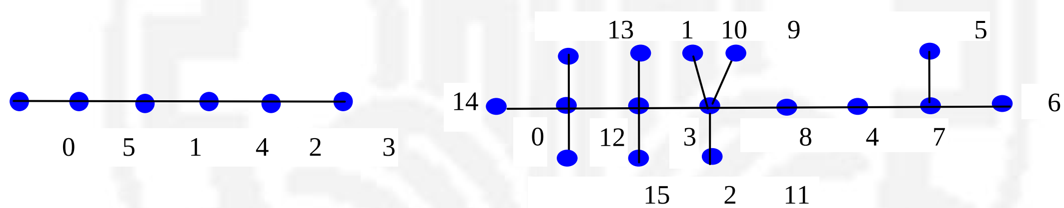


Figure 27: Examples of α-labeling for a snake and a caterpillar

If a tree T is not a caterpillar but Z_T is, then T is called a *lobster*. In 1979 Bermond [21] conjectured that lobsters are graceful. This conjecture is not proved yet but it may be easier to prove than the long intractable Ringle-Kotzig conjecture. Some special cases of Bermond ‘s conjecture were done by Huang and Rosa [55] and Ng [87].

A *symmetrical tree* is a tree consisting of a generator node called the *root* and *t* levels of nodes in such a way that every level contains vertices of same degree. A special symmetrical tree is a *complete k-ary tree* in which the degree of the root is k and the degree of the other vertices except the last level's vertices are k+1. The degree of the last level's vertices are all one. Therefore a complete k-ary tree consists of t levels of nodes has k^{t-1} nodes in the last row. The following theorem was proved in this case:

Theorem 30 [21]: A symmetrical tree is graceful.

As a corollary of the theorem.31 we obtain that a complete k-ary tree is also graceful. A graceful labeling of binary tree is shown in Figure 28:

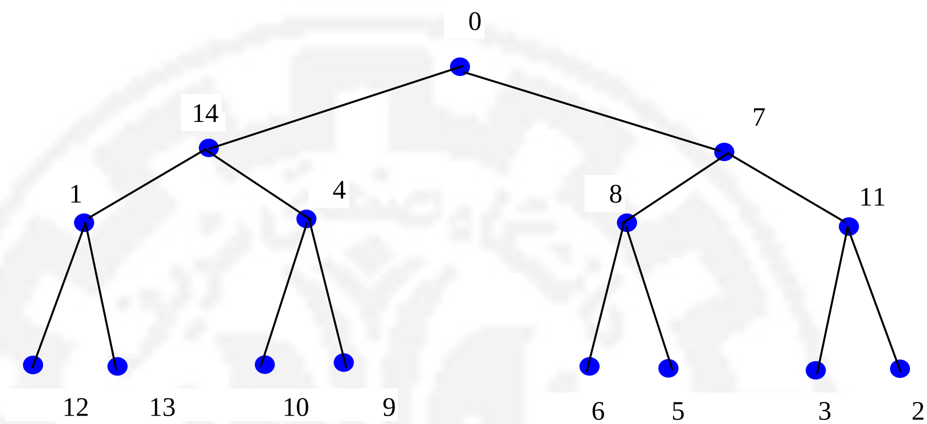


Figure 28: Graceful labeling of a binary tree

As we mentioned there are only a few classes of trees without an α-labeling. The general nonexistence theorem for α-labeling of trees is as follows:

Theorem 31 [57]: Let T = (V,E) be a tree all of whose vertices are of odd degree and m = $|V(T)|$ (thus m ≡ 0 (mod 2)). Let T* be a tree obtained from T by replacing every edge of T by a path of length two. If m ≡ 0 (mod 4) then the tree T* does not have an α-labeling.

The minimal tree belonging to this class of trees is shown in the Figure 29:

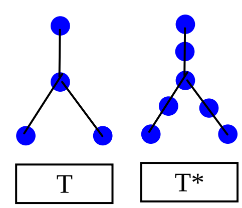


Figure 29: A class of trees with no α-labeling

We know that every snake P_n has an α-valuation. Rosa proved the following theorem about labeling snakes with an additional constraint:

Theorem 32 [91]: Let v be an arbitrary vertex of the snake P_n. Then

a) There exists a graceful labeling Ψ of P_n such that $\Psi(v) = 0$.

b) There exists an α- labeling Ψ of P_n such that $\Psi(v) = 0$ if and only if v is not the central vertex of P_4.

Suppose that we read the labels of P_n from left to right. It is not very difficult to show that the snake P_n admits only one graceful labeling (or α-labeling) whose first label is zero. Frucht and Salinas [37] have described an algorithm to construct all of the graceful labelings of P_n whose first label is one. In order to estimate the number of possible graceful labeling or α-labeling of a snake, Abrham and Kotzig introduced the concept of an α_k-valuation as follows:

Definition 13 [8]: Let $0 \le k \le n$ and P_n be a snake with n edges and with the end vertices w and z. Let Ψ be an α-labeling of P_n. Then Ψ will be called an α_k-labeling (or α_k-valuation) of P_n if $\min(\Psi(w), \Psi(z)) = k$.

Abrham and Kotzig also presented some results concerning the number of α_k-valuation of P_n as follows:

Theorem 33 [8]: Let $N_k(n)$ denote the number of α_k-valuation of P_n, then

1. $N_0(n) =1$ for every $n \ge 1$.
2. $N_1(1) =0$, $N_1(2) =N_1(3) =1$, $N_1(4) =0$, $N_1(n2m) \ge (1/4)\, 2^{[n/3]}$ for all $n \ge 5$.
3. $N_2(n) =0$ for $n =1,2,3,6,8$ and $N_2(n) =1$ for $n = 4,5,7$ and $N_2(n) \ge 2^{[n/3]}$ for $n \ge 9$.

It is shown by theorem 34 that there is an exponential lower bound for the number of α-valuation of the snake P_n. Therefore, the number of graceful valuation of the snake P_n grows at least exponentially with n. On the other hand, we also want to know in which situations P_n has an α_k-valuation for all $k \ge 0$. Abrham proved the following theorem for all pairs n, k for which the snake P_n has an α-valuation:

Theorem 34 [3]: (1) Let n be an odd integer: $n =2m+1 \ge 2k+1$. Then P_n has an α_k-valuation Ψ_k with the end vertices w and z. This α_k-valuation of P_n satisfies the condition $|\Psi(z) - \Psi(w)| = m+1$.

(2) Let n be an even integer: $n = 2m \geq 2k+2$. Then P_n has an α_k-valuation Ψ_k with the end vertices w and z. This α_k-valuation of P_n satisfies the condition $\Psi(z) + \Psi(w) = m$.

For example in Table 2.3 an α_k-valuation of P_{11} for k =0,1,2,3,4,5 and an α_k-valuation of P_{10} for k =0,1,2 are shown: (An α_k-valuation of P_n is described by a sequence of n+1 nonnegative integers in parentheses giving the values of the successive vertices.)

k	α_k-valuation of P_{11}	α_k-valuation of P_{10}
0	(0,11,1,10,2,9,3,8,4,7,5,6)	(0,10,1,9,2,8,3,7,4,6,5)
1	(1,11,0,9,2,10,4,8,3,6,5,7)	(1,9,0,10,3,7,2,8,5,6,4)
2	(2,9,1,10,0,11,5,6,4,7,3,8)	(2,9,1,10,0,6,5,7,4,8,3)
3	(3,8,4,7,5,6,0,11,1,10,2,9)	
4	(4,7,5,6,2,9,3,8,0,11,1,10)	
5	(5,6,4,7,3,8,2,9,1,10,0,11)	

Table 3: α_k-valuation for P_{10} and P_{11}

If we have an α_k-valuation Ψ_k of P_{2m+1} then both the complementary valuation $\Psi_k^{\oplus}$ and the inverse labeling $\Psi_k^{\lozenge}$ are α_{m-k}-valuations of P_{2m+1}. If Ψ_k is an α_k-valuation of P_{2m} then so is $\Psi_k^{\lozenge}$ but if $2k \leq m-1$ then $\Psi_k^{\oplus}$ is an α_{m+k}-valuation of P_{2m} ; if $m \leq k$, $3m > 2k$ then $\Psi_k^{\oplus}$ is an α_{k-m}-valuation of P_{2m}. Examples of these relationships are shown in Figure 30:

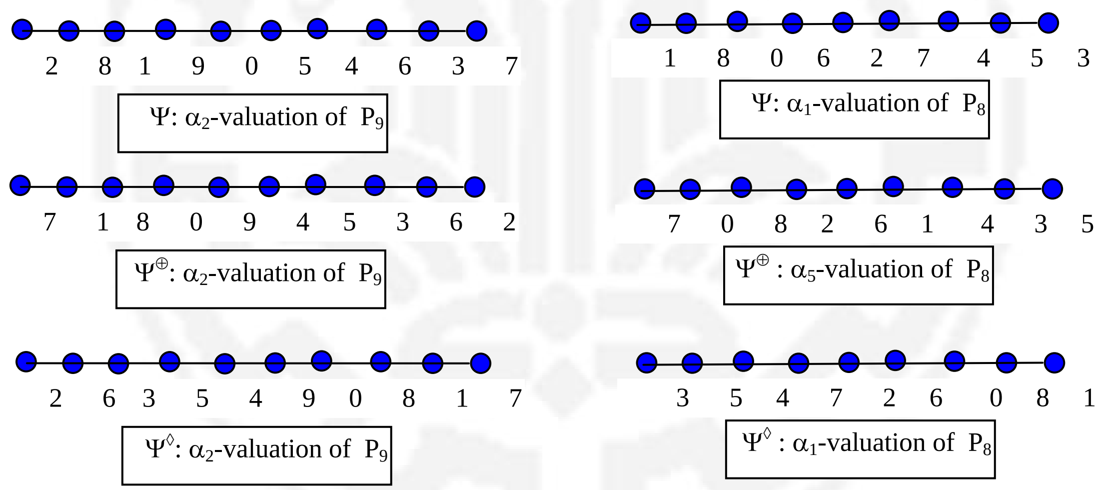


Figure 30: Examples of relationship of α_k-valuation of Ψ and $\Psi^{\oplus}$ and $\Psi^{\lozenge}$

In a number of problems concerning the existence of α-valuations of snakes, the following approach, introduced by Abrham[13], is used: Two snakes with given α-valuations are joined by means of an additional edge, the values of their vertices are suitably transformed, and the result is an α-valuation of a "*longer*" snake.

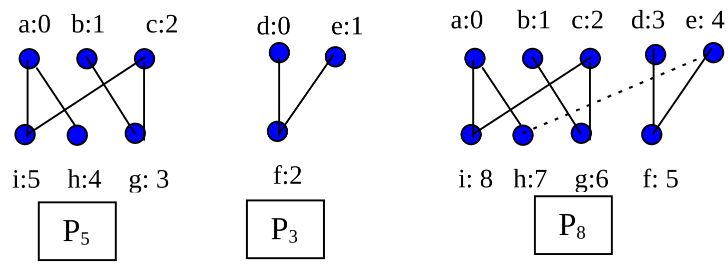


Figure 31: Construction of a "large" snake

For instance, in Figure 31 a construction of an α-valuation of P_8 from the given α-valuations of P_5 and P_3 is shown where $V(P_5) = \{ a, b, c, g, h, i \}$; $V(P_3) = \{ d, e, f \}$and $V(P_8) = \{ a, b, c, d, e, f, g, h, i \}$.

This process can naturally be reversed. It is now time to ask if every α-valuation of "large" snake can be obtained in this way. Unfortunately the answer to this question is negative in general case.

Now let Ψ be an α-valuation of a graph G and let γ be the number from the definition of an α-valuation. Then the sets L(G), U(G) will be defined as follows:

$$L(G) = \{ v \in V(G); \Psi(v) \leq \gamma \}$$
$$U(G) = V(G) - L(G)$$

Definition 14: An α-valuation Ψ of the snake P_n is called *separable* if there exists an edge $e \in E(P_n)$ called a *separator* such that the two graphs Q_1, Q_2 obtained from P_n by deleting e have the following properties:

1. Q_1 and Q_2 are snakes (i.e. they each have at least two vertices).
2. Each of the four sets $\Psi(V(Q_i) \cap L(P_n))$, $\Psi(V(Q_i) \cap U(P_n))$, $i = 1,2$ is either a consecutive integer interval or a one point set.

For example consider an α-valuation of P_8 in Figure 27. We know that $L(P_8) = \{ a, b, c, d, e\}$, $U(P_8) =\{ f, g, h, i \}$. Let us assume that we delete the edge { e, h } and the graphs $Q_1 = P_5$ and $Q_2 = P_3$ obtained from P_8. Then we will have:

$$\Psi(V(Q_1) \cap L(P_8)) = \{ 0,1,2 \} \qquad \Psi(V(Q_1) \cap U(P_8)) = \{ 6,7,8 \}$$
$$\Psi(V(Q_2) \cap L(P_8)) = \{ 3,4 \} \qquad \Psi(V(Q_2) \cap U(P_8)) = \{ 5 \}$$

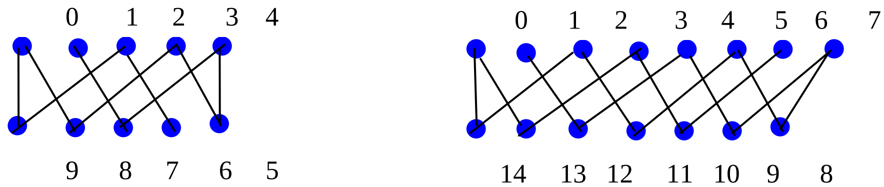
Therefore an α-valuation of P_8 in Figure 31 is separable.

An α-valuation of P_n which is not separable will be called *nonseparable*. We should mention here that, if an α_k-valuation of P_n is separable, the deletion of a separator does not necessarily yield two α_k-valuations of the resulting snakes.

Abrham and Kotzig [13] considered the problem of the existence of nonseparable α-valuation for all snakes. As we have seen before, P_n has exactly one α_0-valuation for any $n \geq 1$. It is easy to verify that these α_0-valuation are separable for $n \geq 3$ and that each edge of P_n not incident with an end vertex is a separator. For α_1-valuations, they have obtained the following result:

Theorem 35 [13]: The snake P_n does not have any nonseparable α_1-valuation if and only if $n \in \{3,5\}$ or $n \equiv 1 \pmod 3$, $n \geq 4$. P_n has exactly one nonseparable α_1-valuation if and only if $n \equiv 0 \pmod 3$, $n \geq 6$, or $n \equiv 2 \pmod 3$, $n \neq 5$.

The nonseparable α_1-valuation of P_9 and P_{11} are shown in Figure 32:



Nonseparable α_1-valuation of P_9 Nonseparable α_1-valuation of P_{14}

Figure 32: Nonseparable α_1-valuations of P_9 and P_{11}

The problem of existence of nonseparable α_2-valuations of P_n has been solved by Abrham and Kotzig but the number of such α_2-valuations is still an open problem:

Theorem 36 [13]: The snake P_n has a nonseparable α_2-valuation if and only if $n \geq 9$.

Combining the Theorems 36 and 37, we obtain:

Theorem 37 [13]: The snake P_n has a nonseparable α-valuation if and only if either $n \in \{1,2,6\}$ or $n \geq 8$.

7. DISJOINT UNION OF GRACEFUL GRAPHS

Definition 15: Given n disjoint graphs $G_1, G_2, \ldots, G_n$ such that they have no vertex or edge in common, the $G_1 \cup G_2 \cup \ldots \cup G_n$ is the graph G with the vertex set and edge set consisting of all those vertices and edges which are in G_1 or G_2 or … or G_n ; symbolically:

$$V(G) = V(G_1 \cup G_2 \cup \ldots \cup G_n) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_n)$$

$$E(G) = E(G_1 \cup G_2 \cup \ldots \cup G_n) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$$

It has been shown in section 2.3.1. that a necessary condition for the existence of a graceful labeling of an Eulerian graph G is as follows:

(NC1) An Eulerian graph G is graceful $\Rightarrow |E(G)| \equiv 0$ or 3 (mod 4)

Furthermore, the existence of an α-valuation for an Eulerian graph can be obtained directly from (NC1) as follows:

(NC2) An Eulerian graph G has an α-valuation $\Rightarrow |E(G)| \equiv 0 \pmod 4$

We know that (NC1) and (NC2) are also sufficient if G is a cycle. In 1996, Abrham proved that (NC1) is also sufficient for 2-regular graphs with two components:

Theorem 38 [6]: Let p, q ≥ 3. Then the graph $C_p \cup C_q$ has a graceful valuation if and only if $p + q \equiv 0$ or 3 (mod 4). $C_p \cup C_q$ has an α-valuation if and only if both p, q are even and $p + q \equiv 0 \pmod 4$.

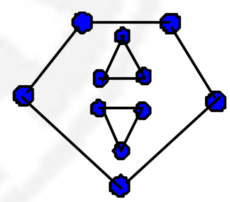


Figure 33: $2C_3 \cup C_5$

According to Theorem 39 the graphs $C_{4k} \cup C_{4m}$ and $C_{4k+2} \cup C_{4m+2}$ have an α-valuation for all k, m ≥ 1 and the graphs $C_{4k} \cup C_{4m-1}$, $C_{4k+2} \cup C_{4m+1}$, and $C_{4k+1} \cup C_{4m+1}$ are only graceful for all k,m ≥1. Kotzig [70] has shown that (NC1) is not sufficient for all 2-regular graph with more than two components. The smallest 2-regular graphs which satisfies (NC1) and is not graceful is the graph $2C_3 \cup C_5$ i.e. the graph G with 11 edges and consisting of two triangle and a pentagon as Figure 33.

Kotzig also proved that a graceful 2-regular graph can not have too many components of odd length, more exactly he proved the following necessary condition for the cases where we have odd cycles in 2-regular graphs:

Theorem 39 [70]: Let w be the number of cycles of odd length in a 2-regular graph G. If G is graceful then $|V(G)| \geq w(w+2)$.

For instance, for the graph in Figure 33 we have $w = 3$ and $|V(G)| = 11$ and $|V(G)| < w(w+2)$ thus according to theorem 2.40 graph G is not graceful.

Abrham has extended the number of nongraceful graphs which satisfy (NC1) by the following theorem:

Theorem 40 [6]: For every $p \geq 11$, $p \equiv 0$ or 3 (mod 4) there exists a 2-regular graph G_p which has p vertices, satisfies (NC1) and is not graceful.

A 2-regular graph G is called an *rC_n graph* if G consists of r cycles and each of them is of length n. In other words, rC_n is the disjoint union of r isomorphic components in such a way that each component is a cycle of length n. In an rC_n graph, we have $|V(rC_n)| = |E(rC_n)| = r \times n$.

Some important results on the gracefulness of this class of graphs are as follows mostly due to Abrham and Kotzig:

1. $1C_n$ (i.e. a cycle with length n) is graceful if and only if $n \equiv 0$ or 3 (mod 4). $1C_n$ also has an α-valuation if and only if $n \equiv 0 \pmod 4$ [90].
2. $2C_n$ is graceful and also has an α-valuation if and only if $n \equiv 0 \pmod 2$ for $n > 2$ [66].
   Note: Results 1 and 2 entirely solve the existence of graceful valuation and also α-valuation of rC_n when $r = 1$ or 2.
3. $3C_{4n}$ has an α-valuation for each $n > 1$. $3C_4$ is graceful but it has no α-valuation [66].
4. $3C_{4n+5}$ is graceful for every $n \geq 1$ [70].
5. $4C_{4n}$ has an α-valuation for all $n \geq 1$ [74].
6. $(r+1)C_3$ and rC_5 have no graceful valuation for $r \geq 1$.C_3 has a graceful valuation [66].
7. rC_4 has an α-valuation for at least $1 \leq r \leq 10$, $r \neq 3$ [70].

8. $(2n\text{-}1)C_{2n+1}$ is not graceful for all $n \geq 1$ [70].
9. r^2C_4 and $(r + r^2)C_4$ have an α-valuation for all $r \geq 1$ [11].
10. If rC_4 has an α-valuation, then $(4r+1)C_4$, $(5r+1)C_4$ and $(9r+2)C_4$ also have an α-valuation [10].

According to results 7,9 and 10 we can conjecture that rC_4 has an α-valuation for all $r \geq 1$, $r \neq 3$. This conjecture was an open problem for years until Abrham and Kotzig proved it in 1994:

Theorem 41 [12]: The graph rC_4 has an α-valuation for all $r \geq 1$, $r \neq 3$.

For example in Figure 34 an α-valuation for $6C_4$ has been shown :

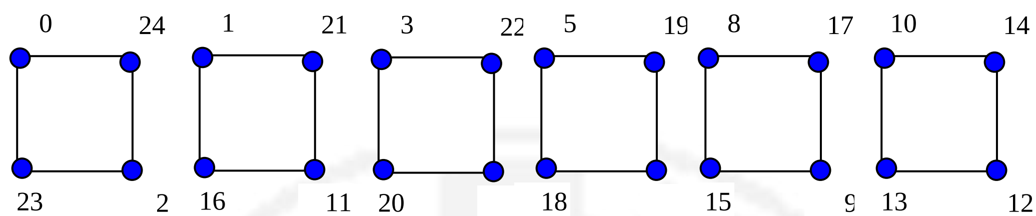


Figure 34: An α-valuation of $6C_4$

Abrham proved the following theorem on 2-regular bipartite graphs to obtain certain rules for extensions of α-valuation generalizing those given in result 10 above:

Theorem 42 [2]: Let $H(4s)$ be a 2-regular bipartite graph on 4s vertices such that the graph $rH(4s)$ has an α-valuation where r, s are positive integers. Then the following graphs also have α-valuations:

1. $4rs\ H(4s) \cup C_{4s}$
2. $[(4s+1)r +1]\ H(4s)$
3. $[(8s+1)r+2]\ H(4s)$ if $2H(4s)$ has an α-valuation

Furthermore if we replace any copy of $rH(4s)$ by another 2-regular bipartite graph on 4rs vertices, which has an α-valuation, the resulting graphs will again have an α-valuations.

For instance suppose $H(4s) = C_8$ i.e. $s = 2$. We know that C_8 has an α-valuation i.e. $r = 1$ thus $8C_8 \cup C_8$ or $9C_8$ has an α-valuation too. Moreover the graphs $10C_8$ and $19C_8$ have an α-valuation according to the parts 2 and 3 of theorem 2.42 and the fact that $2C_8$ has an α-valuation. Any C_8 in these graphs can be replaced by an α-valuation of $2C_4$ and the resulting graphs will again have an α-valuation. This leads to the following statements:

أ. The graph $kC_8 \cup 2(9\text{-}k)C_4$ has an α-valuation for $0 \le k \le 9$.

ب. The graph $kC_8 \cup 2(10\text{-}k)C_4$ has an α-valuation for $0 \le k \le 10$.

ت. The graph $kC_8 \cup 2(19\text{-}k)C_4$ has an α-valuation for $0 \le k \le 19$.

In fact if we put $H(4s) = C_{4n}$ and $r = 1$ in theorem 2.42 since we know that C_{4n} has an α-valuation for $n \ge 1$ we will conclude the following corollary:

Corollary 1: $(4n+1)C_{4n}$, $(4n+2)C_{4n}$ and $(8n+3)C_{4n}$ have an α-valuations.

One of the interesting properties of a graceful labeling of a 2-regular graph is this: If G is a 2-regular graph with a graceful labeling Ψ then there exists a unique integer x ($0 \le x \le |V(G)|$) such that $\Psi(v) \ne x$ for all $v \in V(G)$. If Ψ is an α-valuation of G and $|V(G)| = 4k$ then either $x = k$ or $x = 3k$ [66]. This number x will be referred to as the *missing value*. For instance the graceful labeling and α-labeling of C_8 are shown in the Figure 35. As we can see the missing values of these labelings are $x_1 = 4$ and $x_2 = 2$ respectively:

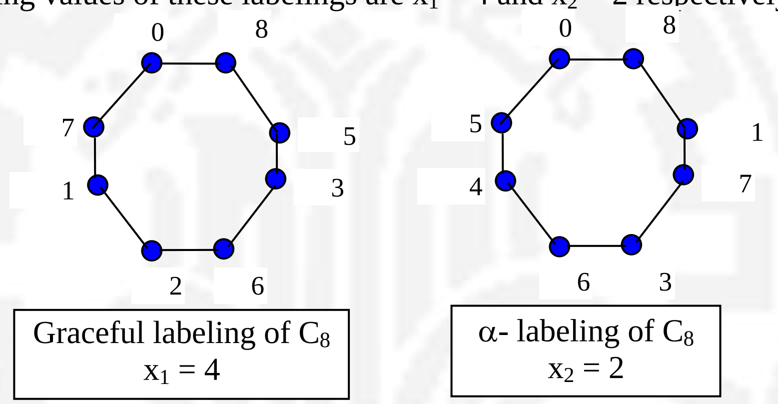


Figure 35: Missing values in graceful labeling of C_8

The following results related to missing value are due to Abrham and Kotzig:

Theorem 43 [7]: Let G be a 2-regular graph on n vertices possessing a graceful labeling Ψ. Then the missing value x has the following properties:

1. If $n = 4k$ then $k \leq x \leq 3k$. Moreover, Ψ is an α-valuation of G if and only if either $x = k$ or $x = 3k$. If Ψ is an α-valuation with $x = k$ (or $x = 3k$), then the complementary valuation $\Psi^{\oplus}$ has an α-valuation with $x^{\oplus} = 3k$ (or $x^{\oplus} = k$).
2. If $n = 4k-1$ then $k \leq x \leq 3k-1$.

A different kind of disjoint union of graphs have been considered by Frucht and Salinas: The union of a snake and a square as follow:

Theorem 44 [37]: If $n \geq 4$ then the graph $C_4 \cup P_n$ is graceful.

In Figure 36 the graceful union of $C_4 \cup P_{13}$ is shown:

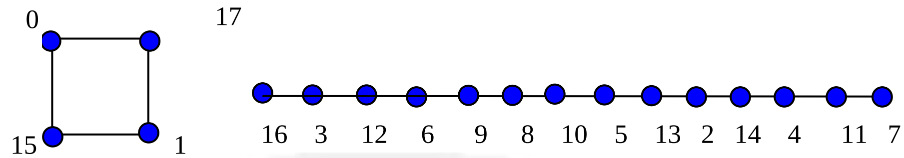


Figure 36: Graceful labeling of $C_4 \cup P_{13}$

Kotzig and Turgeon have studied the graceful valuation of r-regular graphs consisting of m complete graphs. They proved that

Theorem 45 [72]: An r-regular graph consisting of m complete graphs K_{r+1} is graceful if and only if $m = 1$ and $r < 4$.

Graceful valuations of K_1, K_2, K_3 and K_4 are shown in Figure 37:

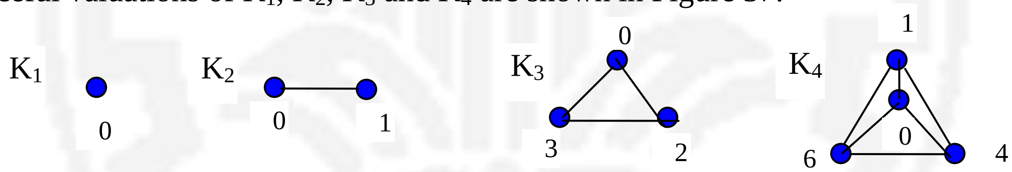


Figure 37: Graceful labeling of K_1, K_2, K_3 and K_4

8. APPLICATIONS OF GRACEFUL GRAPH

Labeled graphs serve as useful tools for a broad range of applications. Bloom and Golomb [26,27] in two excellent surveys have presented systematically an application of graph labeling in many research fields such as coding theory problems, X-ray crystallographic analysis, communication network design, optimal circuit layout, integral voltage generator, and additive number theory. In this section we restrict our discussion to applications of graceful labeling and its variations in decomposition of graphs, perfect system of difference sets, and integer sequences such as the Skolem sequence:

8.1 GRAPH DECOMPOSITION

Definition 16 [29]: A *decomposition of a graph G* is a family H = (H_1,H_2, … , H_n) of subgraphs of G such that each edge of G is contained in exactly one member of H. In fact G is the edge disjoint union of its subgraphs H_i where i = 1,2, … ,n such that $E(H_i) \cap E(H_j) = 0$ for $i \neq j$; $E(G) = \cup E(H_i)$ i = 1,2, … ,n ; $V(G) = \cup V(H_i)$ i = 1,2, … ,n.

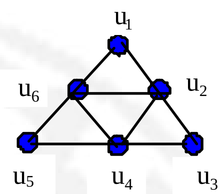


Figure 38 : Decomposition of a graph

For example the graph G shown in Figure 38 has a decomposition H = (H_1, H_2, H_3) into three K_3: $E(H_1)$ = {(u_1, u_2), (u_2, u_6), (u_1, u_6), $E(H_2)$ = {(u_2, u_3), (u_3, u_4), (u_2, u_4), $E(H_3)$ = {(u_1, u_4), (u_1, u_6)} and $V(H_1)$ = (u_1, u_2, u_6), $V(H_2)$ = (u_2, u_3, u_4), $V(H_3)$ = (u_4, u_5, u_6).

Definition 17: Let two graphs G and G′ be given. A *G-decomposition of a graph G′* is a decomposition of G into subgraphs isomorphic to G. In other words, each member H_i in definition 2.16 must be isomorphic to G. We write $G \mid G'$ whenever a G-decomposition of G′ exists.

The decomposition of graph G in Figure 38 is a K_3-decomposition, i.e., $K_3 \mid G$.

Definition 18: A decomposition H of a graph G into subgraphs $H_1, H_2, \ldots, H_n$ is said to be *cyclic* if there exists an isomorphism f of G which induces a cyclic permutation f_v of the set V(G) and satisfies the following implication: if $H_i \in H$ then $f(H_i) \in H$ for $i = 1,2, \ldots ,n$. Here $f(H_i)$ is the subgraph of G with vertex set $\{f(u);\ u \in V(H_i)\}$ and edge-set $\{ (f(u), f(v));\ e = (u, v) \in E(H_i) \}$.

For instance the graph G shown in Figure 39 has a cyclic decomposition with the following permutation:

$$f_v = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ u_2 & u_3 & u_4 & u_5 & u_6 & u_1 \end{pmatrix}$$

The permutation f_v assigns to an element in the first line the element standing below it.

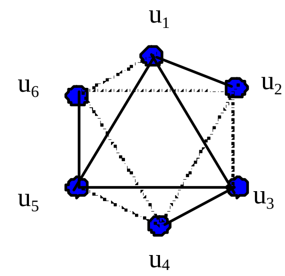


Figure 39 : Cyclic decomposition of a graph

The following theorem explains the connection between an α-valuation and cyclic decomposition of the complete graph into isomorphic subgraphs. This theorem is due to Rosa:

Theorem 46 [90]: If a graph G with n edges has an α-valuation then, for every positive integer c, there exists a cyclic decomposition of the complete graph K_{2cn+1} into subgraphs isomorphic to G.

In other words if G with n edges has an α-valuation then $G \mid K_{2nc+1}$ for $c \geq 1$.

The previous theorems and the results about α-valuation combine to give the following corollary:

Corollary 2: In the cases listed below there exists a cyclic G-decomposition of K_v:

1. $G = C_{4n}$ and $v \equiv 1 \pmod{8n}$
2. $G = P_n$ and $v \equiv 1 \pmod{2n}$
3. $G = K_{n1,\, n2}$ and $v \equiv 1 \pmod{2n_1n_2}$
4. $G = Q_n$ and $v \equiv 1 \pmod{n2^n}$
5. $G = B_{2n}$ and $v \equiv 1 \pmod{12n+2}$
6. $G = Q_n(K_{3,3})$ and $v \equiv 1 \pmod{3(n+2)2^n}$

7. $G = Q_n(K_{4,4})$ and $v \equiv 1 \pmod{(n+3)2^{n+2}}$
8. $G = Q_n(P_k)$ and $v \equiv 1 \pmod{[(n+1)k-2]2^{n-1}}$.

In Figure 40 a cyclic C_4-decomposition of K_9 is shown:

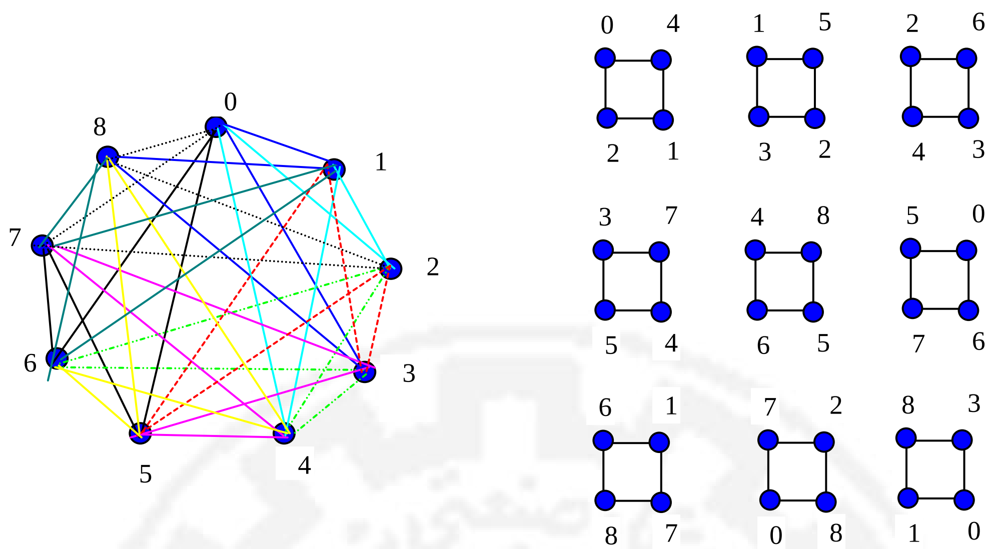


Figure 40: The cyclic $C_4 \mid K_9$

To obtain the cyclic $C_4 \mid K_9$, the vertices of K_9 are labeled with the integers 0,1,2, … ,8. Then consider an α-valuation of C_4. The vertices of C_4 i.e. v_1, v_2, v_3, v_4 in this α-valuation are labeled as $\Psi(v_1) = 0$, $\Psi(v_2) = 4$, $\Psi(v_3) = 1$ and $\Psi(v_4) = 2$. The rest of the cycles C_4 are labeled as follow: the jth cycle of C_4 has the vertices of K_9 labeled $\Psi(v_i) + j-1 \pmod 9$; $i = 1,2,3,4$ and $j = 2,3, \ldots ,9$.

8.2 PERFECT SYSTEM OF DIFFERENCE SETS

Definition 19: Let c, m, $p_1, p_2, \ldots, p_m$ be positive integers, and $S_i = \{ X_{0i} < X_{1i} < \ldots < X_{p_i,i} \}$; $i = 1,2, \ldots ,m$ be a sequence of integers and $D_i = \{ X_{ji} - X_{ki} , 0 \le k < j \le p_i \}$, $i = 1,2, \ldots ,m$ be their difference sets. Then we say that the system $\{ D_1, D_2, \ldots ,D_m \}$ is a *perfect system of difference sets (PSDS)* starting with c if

$$\bigcup_{i=1}^{m} D_i = \{c, c+1, c+2, \dots , c-1 + \sum_{i=1}^{m} (1/2)(p_i (p_i +1)) \}$$

Each set D_i is called *a component of PSDS* $\{ D_1, D_2, \dots , D_m \}$. The size of D_i is p_i. A PSDS is called *regular* if all its components are of the same size i.e. $p_1 = p_2 = \dots = p_m = n-1$. Traditionally a regular PSDS with m components of size n-1 starting at c is referred to as (m, n, c).

If we put $X_{j+k-1,\, i} - X_{j-1,\, i} = d_{ji}(k)$, $j = 1, 2, \dots , p_i +1-k$, $k = 1, 2, \dots , p_i$, $i = 1, 2, \dots , m$, then the elements of D_i can be represented in the form of a difference triangle:

$$\begin{array}{c}
d_{1i}(p_i) \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
d_{1i}(2) \quad d_{2i}(2) \quad \cdots\cdots\cdots\cdots \quad d_{pi-2,i}(2) \quad d_{pi-1,i}(2) \\
d_{1i}(1) \quad d_{2i}(1) \quad d_{3i}(1) \quad \cdots\cdots\cdots \quad d_{pi-2,i}(1) \quad d_{pi-1,i}(1) \quad d_{pi,i}(1) \\
\hline
X_{0i} \quad X_{1i} \quad X_{2i} \quad \cdots\cdots\cdots\cdots\cdots \quad X_{pi-1,i} \quad X_{pi,i}
\end{array}$$

Biraud and Blum and Ribes [5] were probably the first ones to observe a relationship between graceful labeling of graphs and PSDS. The regular PSDS (1,n,1) is a PSDS with one component starting with 1. There exists only two regular PSDS (1,n,1) [5]. They are

$$\begin{array}{ccc}
 & 3 & \\
1 & & 2 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & 6 & & \\
 & 4 & & 5 & \\
1 & & 3 & & 2 \\
\hline
\end{array}$$

$$S:\ 0 \quad 1 \quad 3 \qquad\qquad S:\ 0 \quad 1 \quad 4 \quad 6$$

The mirror images of the above PSDS are also PSDS.

As a matter of fact a PSDS (1,n,1) is related to the graceful labeling of K_n. For instance (1,3,1) and (1,4,1) are the graceful labeling of K_3 and K_4 respectively:

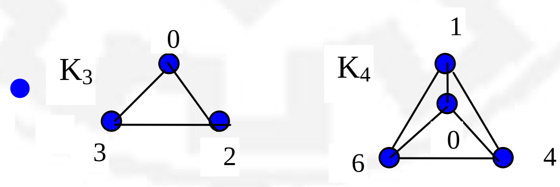


Figure 41: Graceful labeling of K_3 and K_4

In general the existence of a PSDS (m,n,1) corresponds to a graceful labeling of mK_n's having exactly one vertex in common. In section 2.2.2 we have defined this family of graphs as " windmill graphs". Therefore the PSDS of cases (m,3,1) and (m,4,1) reduce to the problem of graceful labeling of Dutch m-windmill and French m-windmill which was discussed in section 2.2.2. Thus there exists a PSDS (m,3,1) if and only if $m \equiv 0$ or 1 (mod 4). A PSDS starting with c describes a c-graceful labeling of a graph which could be decomposed into complete subgraphs. Since there exists no (1,n,c) with c > 1 no complete graph is c-graceful for c >1 (the same as the result obtained in section 2.2.2).
The following regular perfect system (3,3,2)

$$\begin{array}{ccc} \begin{array}{ccc} & 8 & \\ 2 & & 6 \\ \hline 0 & 2 & 8 \end{array} & \begin{array}{ccc} & 9 & \\ 4 & & 5 \\ \hline 0 & 4 & 9 \end{array} & \begin{array}{ccc} & 10 & \\ 3 & & 7 \\ \hline 0 & 3 & 10 \end{array} \\ S_1 & S_2 & S_3 \end{array}$$

implies that the Dutch 3-windmill below is 2-graceful :

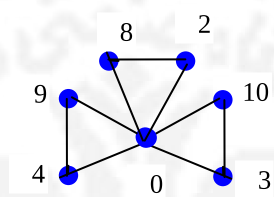


Figure 42: The graceful labeling of Dutch 3-windmill

Note that the same PSDS could generate the k-graceful labeling for different kinds of graphs. For example if we choose the same PSDS as the above but with a different S_2 we will have

$$\begin{array}{ccc} \begin{array}{ccc} & 8 & \\ 2 & & 6 \\ \hline 0 & 2 & 8 \end{array} & \begin{array}{ccc} & 9 & \\ 4 & & 5 \\ \hline 1 & 5 & 10 \end{array} & \begin{array}{ccc} & 10 & \\ 3 & & 7 \\ \hline 0 & 3 & 10 \end{array} \\ S_1 & S_2 & S_3 \end{array}$$

We will find the 2-graceful labeling of Δ_3-snake as Figure 43:

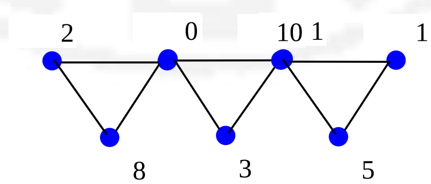


Figure 43: 3-graceful valuation of Δ_3-snake

As we have seen before, Kotzig and Turgeon [72] have proved that graph G consisting of m components where each component is K_r is graceful if and only if m = 1 and r = 2 or 3. The PSDS given below shows that this statement does not hold if the components of G are complete graphs but G is not regular:

```
              16
           14      15
        10      13     6
      1      9      4    2
     -----------------------------        -------------------        ----------------
S1: 0    1      10    14   16       S2: 2     5     13        S3: 3     8     15
                                                11                         12
                                             3       8                  5       2
```

The corresponding graph to this PSDS is graceful labeling of $K_5 \cup 2K_3$ as follows:

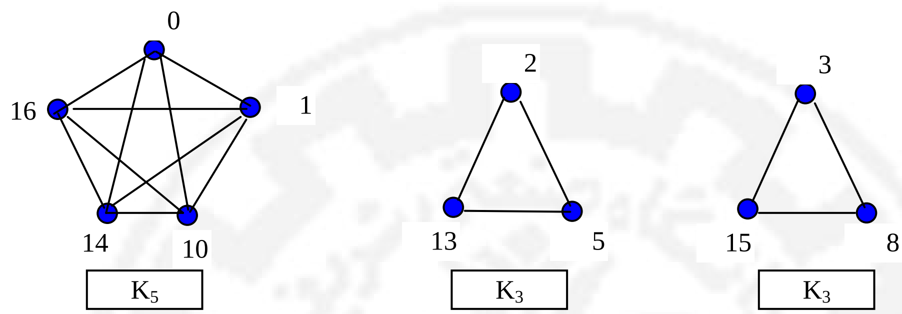


Figure 44: Graceful labeling of $K_5 \cup 2K_3$

8.3 INTEGER SEQUENCES

A graceful graph or its variations can be represented by a sequence of positive integers. Sheppard [94] was the first one to establish a relation between the integer sequences and graceful labelings of graphs.

Definition 20 [94]: For a positive integer m, the sequence of integers $(j_1, j_2, j_3, \ldots, j_m)$ denoted by (j_i), is a *labeling sequence* if and only if $0 \le j_i \le m-i$ for all $i \in [1,m]$.

For instance for m = 5, the sequences (4,3,2,1,0), (0,2,1,1,0), and (1,3,0,1,0) are labeling sequences.

Theorem 47 [94]: There is a one-to-one correspondence between graceful graphs with m edges and labeling sequences (j_i) of m terms.

Let G be a graph with m edges and a graceful labeling Ψ. Then let j_i be the smaller of the end labels of the edge labeled i. In other words $j_i = \min(\Psi(u), \Psi(v))$ $i \in [1,m]$, u,v are the ends of the edge labeled i. Conversely, given a labeling sequence (j_i) with m terms, the graceful labeling can constructed as follows: Arbitrarily assign the m+1 labels of [0,m] to m+1 isolated vertices. For each j_i, join the vertices with the labels j_i and $j_i + i$.

Figure 45 gives all the graceful labelings of a graph with 3 edges, paired with the corresponding labeling sequence:

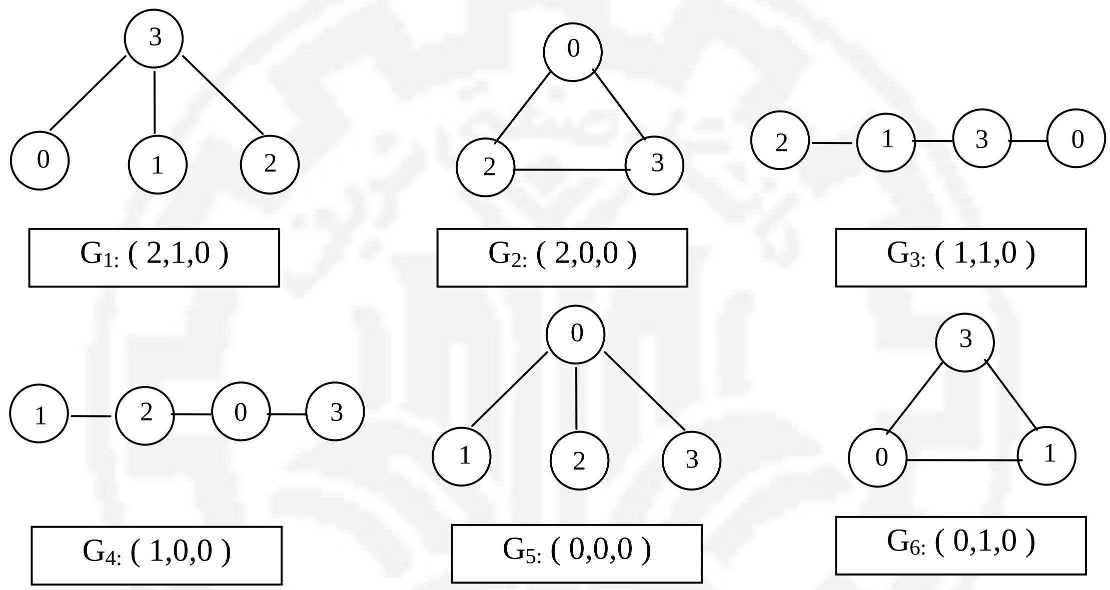


Figure 45: All graceful graphs with 3 edges

Since there are m! labeling sequences with m terms, there are m! graceful graphs with m edges. Some of these graceful graphs have an α-labeling too. If G is a graph with an α-labeling, the corresponding labeling sequence is called a *balanced sequence* and has the following property:

Theorem 48 [94]: The labeling sequence (j_i) with m terms is a balanced sequence if and only if the sequence (j_i^*) defined by $j_i^* = j_1 - j_{m-i+1}$ for all $i \in [1,m]$ is a labeling sequence.

For example in Figure 45 the labeling sequence of G_2 is $(j_{G2}) = (2,0,0)$. Since $(j_{G2}^*) = (2,2,0)$ is not a labeling sequence then (j_{G2}) is not a balanced labeling sequence and G_2 has no α-labeling.

By using the concept of balanced sequence, Sheppard could successfully calculate the number of graphs having an α-labeling as follows:

Theorem 49 [94]: The number of balanced sequences with m terms is

(1) $$2\sum_{j=1}^{(1/2)m} (j!)^2 j^{m-2j} \qquad \text{if m is even}$$

(2) $$2\sum_{j=1}^{(1/2)(m-1)} [(j!)^2 j^{m-2j}] + [((1/2)(m+1))!\,((1/2)(m-1))!] \qquad \text{if m is odd}$$

The number of graphs having graceful and α-labeling for graphs with m edges and their ratios in compare to each other are shown in Table 2.4. As we can see when the number of edges grows, the fraction of graphs having an α-labeling among the graceful graphs approaches near zero:

m	**# of graceful graphs (1)**	**# of graphs having an α-labeling (2)**	**ratio(2) / (1)**
1	1	1	1
2	2	2	1
3	6	4	0.68
4	24	10	0.42
5	120	30	0.25
10	3628800	53578	0.015
15	$1.3 * 10^{12}$	$8.9 * 10^{8}$	$6.8 * 10^{-4}$
20	$2.4 * 10^{18}$	$6.9 * 10^{13}$	$2.8 * 10^{-5}$
30	$2.6 * 10^{32}$	$1.1 * 10^{25}$	$4.2 * 10^{-8}$

Table 4: The number of graphs having graceful and α-labeling

Abrham [1] has studied the relation of graceful labeling of certain regular graphs and another integer sequence referred to as a Skolem sequence:

Definition 21 [96]: A *Skolem sequence of order n* is a sequence $S = \{ S_1, S_2, \ldots, S_{2n}\}$ of positive integers with the following properties:

1. For any $k \in \{ 1, 2, \ldots, n \}$ there exists precisely two subscripts i (k), j (k) such that $S_{i(k)} = S_{j(k)} = k$.

2. The two subscripts satisfy the condition $|i(k) - j(k)| = k$, $k = 1,2, \dots ,m$.

For instance the set $S = \{1, 1, 3, 4, 5, 3, 2, 4, 2, 5\}$ is a Skolem sequence of order 5 because $S_1 = S_2 = 1$, $S_7 = S_9 = 2$, $S_3 = S_6 = 3$, $S_4 = S_8 = 4$, $S_5 = S_{10} = 5$.

Skolem proved the following theorem:

Theorem 50 [96]: A Skolem sequence of order n exists if and only if $n \equiv 0$ or $1 \pmod 4$.

Now suppose that G is a graceful 2-regular graph on n vertices. We want to assign an integer sequence $S(G) = \{ a_0, a_1, \dots , a_n, b_0, b_1, \dots ,b_n \}$to this graceful labeling. Abrham [1] developed the following algorithm for constructing S(G):

Algorithm for constructing S(G)

Note: We assume that the edges of G will be numbered $e_1, e_2, \dots , e_n$ in such a way that the value of e_k in the graceful labeling is k, $k = 1, 2, 3, \dots ,n$.
At a given stage of construction of the terms of S(G) we say that a term (either a_i or b_i) of S(G) is free if it has not been assigned to a value yet.

أ. Select an arbitrary cycle C of G and a direction in which we will move around C.

ب. Choose an arbitrarily edge e_k of C with end vertices having the values i, i+k.

ت. Choose one of the pairs (a_i, a_{i+k}),(b_i, b_{i+k}),(a_i, b_{i+k}),(a_{i+k}, b_i) and assign the value k to both of its terms.

ث. Move to the edge adjacent to e_k at its end point in the direction chosen. Denote that as e_r.

ج. If we consider the edge e_r with the end vertices having the values p, p+r ; we select one of the pairs (a_p, a_{p+r}), (b_p, b_{p+r}),(a_{p+r} , b_p), (a_p, b_{p+r}) which has two free terms (such a choice is possible according to theorem in [1]) and assign the value r to both terms in this pairs.

ح. Continue with all edges of the cycle C that have not been used.

خ. If G has only one cycle, at the end we will be left with one pair (a_x, b_x) containing two free terms; we will then put $a_x = b_x = n+1$. If G has more than one cycle we take another cycle of G and repeat the procedure, until we end with only one free pair (a_x, b_x); then we put $a_x = b_x = n+1$.

Example 2.1: The graph G and its graceful labeling are shown in Figure 46. We want to construct S(G) by choosing a clockwise direction:

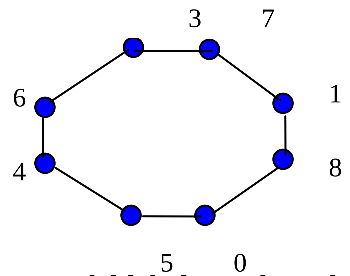


Figure 46: Graceful labeling of graph $G = C_8$

By applying the above algorithm to the graph G in Figure 46, we will obtain the following result as Table 2.5:

stage	elected edge	Possible pairs with two free terms	elected terms
0	4	(a_3,a_7), (a_3,b_7), (b_3,a_7), (b_3,b_7)	$b_3 = b_7 = 4$
1	6	(a_1,a_7), (a_7,b_1)	$a_1 = a_7 = 6$
2	7	(b_1,b_8), (a_8,b_1)	$b_1 = b_8 = 7$
3	8	(a_0,a_8), (a_8,b_0)	$a_0 = a_8 = 8$
4	5	(b_0,b_5), (a_5,b_0)	$b_0 = b_5 = 5$
5	1	(a_4,a_5), (a_5,b_4)	$a_4 = a_5 = 1$
6	2	(b_4,b_6), (a_6,b_4)	$b_4 = b_6 = 2$
7	3	(a_3,a_6), (a_6,b_3)	$a_3 = a_6 = 3$

Table 2.5: Construction of S(G) for graph G in Figure 46

Finally the pair (a_2,b_2) is left. Thus $a_2 = b_2 = n+1 = 9$ and S(G) has the following sequence:

$S(G) = (a_0, a_1, \dots, a_8, b_0, b_1, \dots, b_8) = (8, 6, 9, 3, 1, 1, 3, 6, 8, 5, 7, 9, 4, 2, 5, 2, 4, 7)$.

The sequence S(G) constructed above does not have to be a Skolem sequence, but in two special cases it generates a Skolem sequence of order n+1 by a slight modification of the above algorithm [1]:

1. If G is a 2-regular graceful graph on n vertices, consisting only of cycles of even length. Then $n \equiv 0 \pmod 4$.

2. If G is a 2-regular graph on $n \equiv 3 \pmod 4$ vertices with a single component of odd length.

Conversely, sometimes a Skolem sequence can generate a graceful labeling or α-labeling of a 2-regular graph. The special cases are as follows [1]:

i) Let $S(G) = \{ S_1, S_2, \dots , S_{2n+2} \}$ be a Skolem sequence of order n+1. Furthermore if $S_i = S_{i+k} = k$ for $1 \le k \le n$ and either $i + k \le n+1$ or $i \ge n+2$; then S(G) generates a graceful labeling of a 2-regular graph G on $n \equiv 0 \pmod 4$ vertices consisting one or more cycles of even length.

ii) Let $S(G) = \{ S_1, S_2, \dots , S_{2n+2} \}$ be a Skolem sequence of order n+1, $n \equiv 0 \pmod 4$ and $k \in \{ 1,2, \dots ,n \}$ and $S_i = S_{i+k} = k$:

1. For $i + k \le n+1$, if $i \le (n/2)+1$ then $(n/2) +1 < i + k \le n+1$. For $i \ge n+2$, if $n+2 \le i \le (3n/2) +2$ then $(3n/2) < i + k \le 2n+2$. In this case S(G) generates an α-labeling of 2-regular graph G on $n \equiv 0 \pmod 4$ vertices consisting one or more cycles of even length with $x = (n/4)$, $\gamma = (n/2)$.
2. For $i + k \le n+1$, if $i \le (n/2)$ then $(n/2) < i + k \le n+1$. For $n+2 \le i$, if $n+2 \le i \le (3n/2) + 1$ then $(3n/2)+1 < i + k \le 2n+2$. In this case S(G) generates an α-labeling of 2-regular graph G on $n \equiv 0 \pmod 4$ vertices consisting one or more cycles of even length with $x = (3n/4)$, $\gamma = (n/2)-1$.

iii) Let $S(G) = \{ S_1, S_2, \dots , S_{2n+2} \}$ be a Skolem sequence of order n+1, $n \equiv 3 \pmod 4$ with the following properties:

(1) If $1 \le k \le n$, $k \ne (n+1)/2$, and if $S_i = S_{i+k} = k$ then either $i \ge n+2$ or $i+k \le n+1$.

(2) If $S_i = S_{i+(n+1)/2} = (n+1)/2$ then $i \le n+1$, $i+(n+1)/2 \ge n+2$.

(3) If $S_i = S_{i+n+1} = n+1$ then $i \le n+1$, $i+n+1 \ge n+2$.

In this case S(G) generates a graceful labeling of 2-regular graph G on $n \equiv 3 \pmod 4$ vertices with a single component of odd length containing $e_{(n+1)/2}$.

Example 2.2: $S(G_1) = (8, 5, 9, 4, 1, 1, 5, 4, 8, 6, 7, 9, 2, 3, 2, 6, 3, 7)$ is a Skolem sequence of order 9. $S(G_1)$ satisfies the conditions of part b.1, therefore $S(G_1)$ generates an α-valuation of 2-regular graph on eight vertices.

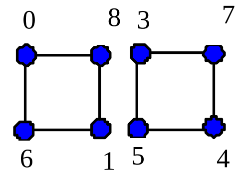


Figure 47 : An α-valuation of $2C_4$

In fact $S(G) = (a_0,a_1, \ldots ,a_8, b_0,b_1, \ldots ,b_8) = (8, 5, 9, 4,1, 1, 5, 4, 8, 6, 7, 9, 2, 3, 2, 6, 3, 7)$ is an α-labeling of $2C_4$ as we can see in Figure 47.

$S(G_2) = (11, 9, 1, 1, 3, 4, 12, 3, 6, 4, 9, 11, 8, 10, 6, 7, 5, 2, 12, 2, 8, 5, 17, 10)$ satisfies the properties c, then $S(G_2)$ generates a graceful valuation of 2-regular graph $C_7 \cup C_4$ as follows:

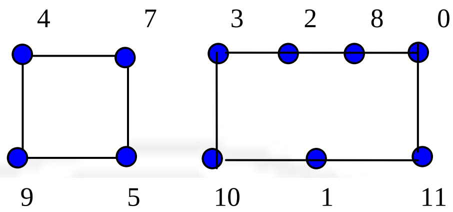


Figure 48: Graceful labeling $C_4 \cup C_7$

Example 3: The Skolem sequences of order $n + 1 = 9$ are easy to enumerate. The enumeration of all Skolem sequence of order 9 yields six α-valuations of C_8, eighteen graceful labelings of C_8 which are not α-valuations, two α-valuations of the 2-regular graph consisting of two 4-cycles, and finally four graceful labeling of this graph which are not α-valuations.

Unfortunately the correspondence between graceful labeling of certain 2-regular graphs and certain Skolem sequence is not one-to-one: A change in orientation of a cycle of the graph changes the resulting Skolem sequence. Nevertheless, this correspondence might in future help to find estimates for the number of graceful numberings of 2-regular graphs, perhaps along the lines used in [4].

8.4 RADAR PULSE CODES

The problem of graceful complete graph K_m is equivalent to the problem of putting m marks on the ruler (always including the ruler's two ends as marks) so that every distance between a pair of marks is a distinct integer. Figure 49 represents the ruler model of K_4 with the vertex values 0, 1, 4 and 6. It can be imagined as a ruler of length 6 with four slots (0, 1, 4, 6) that can be used to measure any integral distance less than or equal to 6.

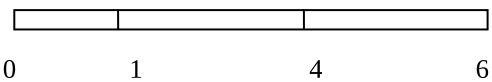


Figure 49: Ruler model of graceful graph K_4

By this process the $\binom{m}{2}$ distances which the ruler can measure are numerically equal to the edge numbers of K_m. It has been previously shown that no complete graph with more than four vertices can be gracefully numbered. Golomb [45] published a generalization of this problem as follow:

Problem 2: Let us assign m distinct non-negative integers to the m vertices of graph $G = (V, E)$, $n = |E(G)|$ and $m = |V(G)|$ in such a way that the n edges receive n distinct positive integers by the assignment of $|a_i - a_j|$ to a given edge, where a_i and a_j are the numbers assigned to its end points. Moreover, we wish to minimize the value of the largest integer assigned to any vertex of G. We call this minimized value $\theta(G)$.
The problem is to assign integers to the vertices of G so as to achieve $\theta(G)$.

It is clear that $\theta(G) \geq n$. A graph for which $\theta(G) = n$ is a graceful graph. A survey of results on this problems can be found in [26, 29, 45, 58]. Here we concentrate on the case of $G = K_m$. In Figure 50 a solution of this problem for $G = K_5$ has been shown. As we can see in this case $\theta(K_5) = 11$ and no edge is numbered 6:

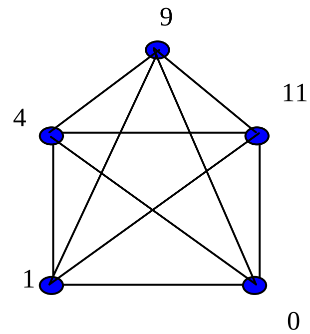


Figure 50 : Numbering of K_5 with θ (K_5) = 11

It follows that rulers corresponding to numberings of K_m, $m \geq 5$, must be longer than $\binom{m}{2}$ if want that no measurements are repeated. Gardner [44] called these m-mark, non-redundant, minimum-length rulers as *Golomb Rulers*. Golomb rulers with fewer than 10 marks have been shown in Table 6:

m	$\binom{n}{2}$	$\theta(K_m)$	**Marks at**
2	1	1	0, 1
3	3	3	0, 1, 3
4	6	6	0, 1, 4, 6
5	10	11	0, 1, 4, 9, 11 or 0, 2, 7, 8, 11
6	15	17	0, 1, 4, 10, 12, 17 or 0, 1 ,4, 10, 15, 17 or 0, 1, 8, 11, 13, 17 or 0, 1, 8, 12, 14, 17
7	21	25	0, 1, 4, 10, 18, 23, 25 or 0, 1, 7, 11, 20, 23, 25 or 0, 1, 11, 16, 19, 23, 25 or 0, 2, 3, 10, 16, 21, 25
8	28	34	0, 1, 4, 9, 15, 22, 32, 34
9	36	44	0, 1, 5, 12, 25, 27, 35, 41, 44
10	45	55	0, 1, 6, 10, 23, 26, 34, 41, 53, 55

Table 6: Golomb rulers for $2 \leq m \leq 10$

Radar distance ranging is accomplished by transmitting a train of pulse and waiting for its return. Because of the dispersion of energy occurring both during transmission of the signal and its scattering during reflection, only a small fraction of the transmitted energy ever returns to the detector. It is desirable to have a very narrow transmitted radar pulse whose instant of return can accurately determined.

If a series of m radar “pulses“ are transmitted corresponding to marks on a non-redundant ruler, it is easy to determine precisely when the pulse train returns.

A signal of relative amplitude m will be generated when the returning signal precisely align with an array of detectors distributed like a template of a transmitted pulse train.

At any other time, no more than one pulse can excite any detector in the template. Moreover, if the temporal positions of the pulses occur at marks positions on a Golomb Ruler, the overall duration of the train will be minimized [26]. Figure 51 Shows a returning pulse-train and the associated detector array, as well as the autocorrelation function of the pulse-train:

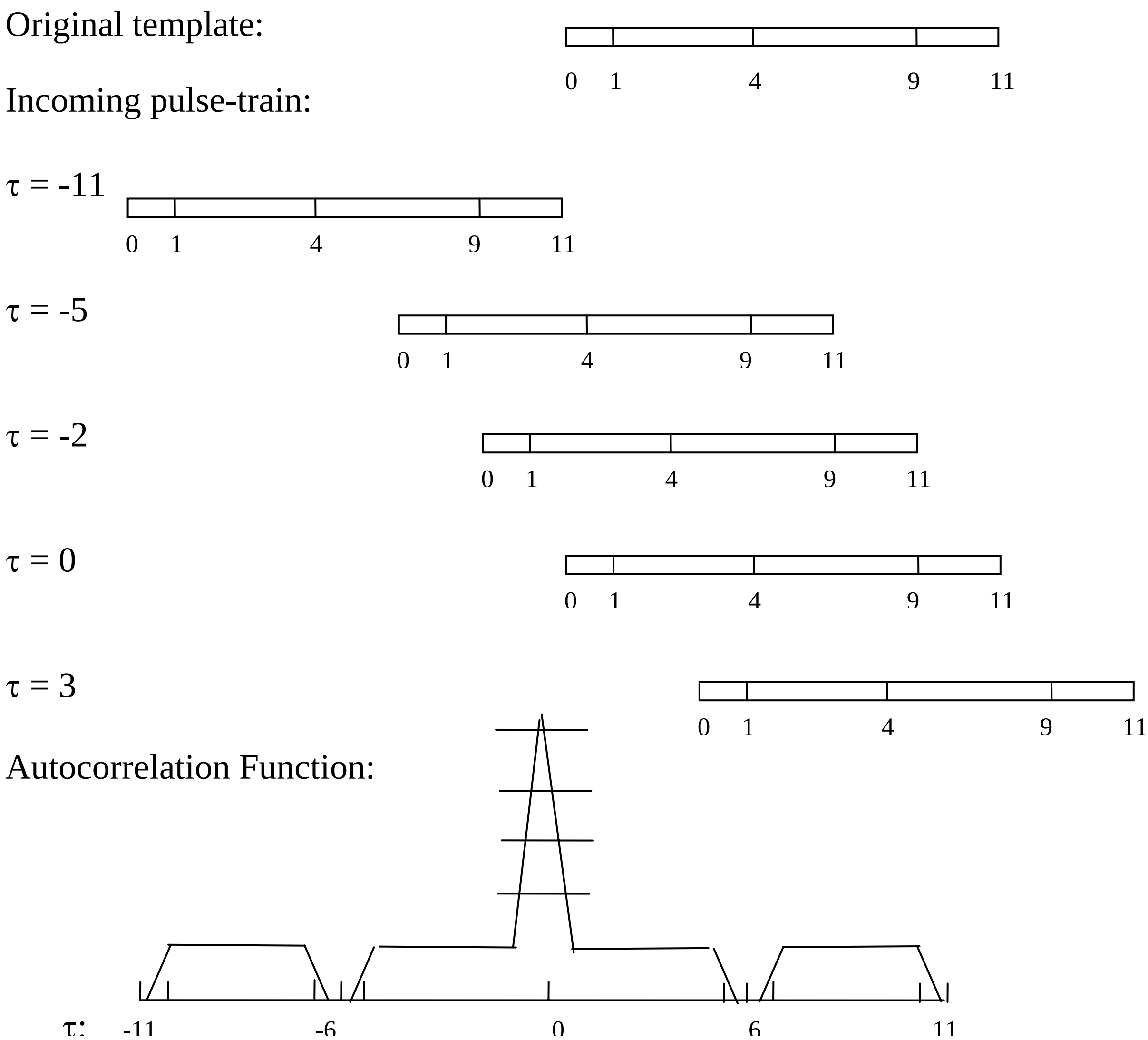


Figure 51: The correlation of a radar code pulse train with an image of itself

Let each pulse be of one unit duration. Thus, when an incoming string matches the original template there can be at most one incoming pulse. In the absence of noise, then, the unnormalized out-of-synch autocorrelation can attain a maximum of 1. A dip in the autocorrelation occurs in ±6 time units, since there are no pulses which are aligned with a six-unit shift of the pulse sequence out of its synch position. Six, of course, is the only distance of 11 units that the original Golomb ruler could not measure and the only numbering missing in numbering of K_5. For further applications of this type of problem see [26, 27].

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